

NOTES ON ERROR ESTIMATES FOR GALERKIN APPROXIMATIONS OF THE ‘CLASSICAL’ BOUSSINESQ SYSTEM AND RELATED HYPERBOLIC PROBLEMS.

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ABSTRACT. We consider the ‘classical’ Boussinesq system in one space dimension and its symmetric analog. These systems model two-way propagation of nonlinear, dispersive long waves of small amplitude on the surface of an ideal fluid in a uniform horizontal channel. We discretize an initial-boundary-value problem for these systems in space using Galerkin-finite element methods and prove error estimates for the resulting semidiscrete problems and also for their fully discrete analogs effected by explicit Runge-Kutta time-stepping procedures. The theoretical orders of convergence obtained are consistent with the results of numerical experiments that are also presented.

1. INTRODUCTION

In this paper we will analyze Galerkin approximations to the so-called ‘classical’ Boussinesq system

$$\begin{aligned}\eta_t + u_x + (\eta u)_x &= 0, \\ u_t + \eta_x + uu_x - \frac{1}{3}u_{xxt} &= 0,\end{aligned}\tag{1.1}$$

which is an approximation of the two-dimensional Euler equations of water-wave theory that models two-way propagation of long waves of small amplitude on the surface of an ideal fluid in a uniform horizontal channel of finite depth. The variables in (1.1) are nondimensional and unscaled; x and t are proportional to position along the channel and time, respectively, and $\eta = \eta(x, t)$ and $u = u(x, t)$ are proportional to the elevation of the free surface above a level of rest represented by $\eta = 0$, and to the depth-averaged mean horizontal velocity of the fluid.

The system (1.1) is a member of a general family of Boussinesq systems derived in [11] that are approximations to the Euler equations of the same order as (1.1) and whose nonlinear and dispersive terms are of equal importance when written in scaled form. These systems are written as

$$\begin{aligned}\eta_t + u_x + (\eta u)_x + au_{xxx} - b\eta_{xxt} &= 0, \\ u_t + \eta_x + uu_x + c\eta_{xxx} - du_{xxt} &= 0,\end{aligned}\tag{1.2}$$

where a, b, c, d are real parameters satisfying $a + b = \frac{1}{2}(\theta^2 - 1/3)$, $c + d = \frac{1}{2}(1 - \theta^2)$, where $0 \leq \theta \leq 1$. The specific system (1.1) has been previously formally derived from the Euler equations, in the appropriate parameter regime, in [24], [26], and [31]. It has been widely used in the engineering fluid mechanics literature for computations of long, nonlinear dispersive waves, starting with [24] and [25].

Existence and uniqueness of the initial-value problem for (1.1) posed for $x \in \mathbb{R}$ and $t \geq 0$ and supplemented by initial conditions of the form

$$\eta(x, 0) = \eta_0(x), \quad u(x, 0) = u_0(x), \quad x \in \mathbb{R},\tag{1.3}$$

was studied by Schonbeck, [27] and Amick, [2]. In these papers global existence and uniqueness was established for infinitely differentiable initial data of compact support such that $\eta_0(x) > -1$, $x \in \mathbb{R}$. In [12] the theory of [27] and [2] was used to prove that given initial data $(\eta_0, u_0) \in H^s(\mathbb{R}) \times H^{s+1}(\mathbb{R})$ for $s \geq 1$, such that $\inf_{x \in \mathbb{R}} \eta_0(x) > -1$, there is a unique solution (η, u) which, for any $T > 0$, lies in $C(0, T; H^s(\mathbb{R})) \times C(0, T; H^{s+1}(\mathbb{R}))$. (Here, $H^s(\mathbb{R})$ is the usual, L^2 -based Sobolev space of functions on \mathbb{R} and $C(0, T; X)$ denotes the space of functions $\phi = \phi(t)$ having, for each $t \in [0, T]$, values in a Banach space X and are such that the map $[0, T] \mapsto \|\phi\|_X$ is continuous.)

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It is well known that the initial-value problem (1.1), (1.3) has classical *solitary-wave* solutions. In [4] we review the properties of such solutions, we construct them numerically and investigate, by means of numerical experiments, the resolution of general initial profiles into sequences of solitary waves, the details of the interaction of solitary waves during head-on and overtaking collisions, the interaction of solitary waves with boundaries, and several other issues. These numerical experiments were performed in [4] by fully discrete Galerkin methods that approximate solutions of (1.1) posed on finite intervals and subject to boundary conditions. It is of interest therefore to study initial-boundary-value problems for (1.1) and establish error estimates for their numerical approximations.

In this paper we shall analyze the numerical solution of the following initial-boundary-value problem (ibvp) for (1.1): For some $0 < T < \infty$, we seek $\eta = \eta(x, t)$, $u = u(x, t)$, defined for $0 \leq x \leq 1$, $0 \leq t \leq T$, and satisfying

$$\begin{aligned} \eta_t + u_x + (\eta u)_x &= 0, & (x, t) &\in [0, 1] \times [0, T], \\ u_t + \eta_x + uu_x - \frac{1}{3}u_{xxt} &= 0, & (x, t) &\in [0, 1] \times [0, T], \\ \eta(x, 0) &= \eta_0(x), \quad u(x, 0) = u_0(x), & x &\in [0, 1], \\ u(0, t) &= 0, \quad u(1, t) = 0, & t &\in [0, T]. \end{aligned} \tag{CB}$$

This ibvp (for $0 < t < \infty$) has been studied by Adamy, [1], who showed that it has weak (distributional) solutions $(\eta, u) \in L^\infty(\mathbb{R}^+; L^1 \times H_0^1)$ provided e.g. that $\eta_0 \in L^1$, $u_0 \in H_0^1$ with $\inf_{x \in [0, 1]} \eta_0(x) > -1$. (Here $L^1 = L^1(0, 1)$, and $H_0^1 = H_0^1(0, 1)$ is the subspace of the Sobolev space $H^1(0, 1)$ whose elements vanish at $x = 0$ and $x = 1$.) The proof uses a parabolic regularization of the first pde of (CB), a technique used in the context of the Cauchy problem by Schonbeck, [27]. It should be noted that the homogeneous Dirichlet boundary conditions on u in (CB) are one kind of boundary conditions that lead to well posed ibvp's in the case of the linearized system, [21]. It is noteworthy that the CB system needs only two boundary conditions for well-posedness as opposed to the four boundary conditions (for example, Dirichlet conditions on η and u at each endpoint of the interval) required in the case of other Boussinesq systems, such as the BBM-BBM ($a = c = 0$, $b = d$ in (1.2), [10]), or the Bona-Smith systems ($a = 0$, $b = d > 0$, $c < 0$ in (1.2), [7].) The case of the homogeneous Dirichlet boundary conditions in (CB) may be viewed as a base for studying the nonhomogeneous analog wherein $u(0, t)$ and $u(1, t)$ are given functions of $t \geq 0$ corresponding to measurements of the velocity variable at two points along the channel.

In [13], Bona, Colin, and Lannes introduced another type of Boussinesq systems that they called ‘completely symmetric’. These are obtained by a nonlinear change of variables from the usual systems (1.2), and have certain mathematical and modelling advantages over the latter. In this paper we shall also therefore consider the analogous problem for the symmetric system, specifically the ibvp

$$\begin{aligned} \eta_t + u_x + \frac{1}{2}(\eta u)_x &= 0, & (x, t) &\in [0, 1] \times [0, T], \\ u_t + \eta_x + \frac{3}{2}uu_x + \frac{1}{2}\eta\eta_x - \frac{1}{3}u_{xxt} &= 0, & (x, t) &\in [0, 1] \times [0, T], \\ \eta(x, 0) &= \eta_0(x), \quad u(x, 0) = u_0(x), & x &\in [0, 1], \\ u(0, t) &= 0, \quad u(1, t) = 0, & t &\in [0, T]. \end{aligned} \tag{SCB}$$

It is not hard to see that the solution of (SCB) satisfies the conservation property

$$\|\eta(t)\|^2 + \|u(t)\|^2 + \frac{1}{3}\|u_x(t)\|^2 = \|\eta_0\|^2 + \|u_0\|^2 + \frac{1}{3}\|u'_0\|^2, \tag{1.4}$$

for $t \geq 0$, which simplifies the study of its well-posedness and the analysis of its numerical approximations as will be seen in the sequel.

In this paper we shall analyze semidiscrete and fully discrete Galerkin-finite element methods for the ibvp's (CB) and (SCB) assuming that their solutions are sufficiently smooth. Previously, rigorous error estimates for Galerkin methods for these ‘classical’ Boussinesq systems were proved in the case of the *periodic* initial-value problem in [3] and [9]. (For numerical work for this type and for other Boussinesq systems of the form (1.2) we refer the reader e.g. to [3], [6], [5], [22], [23], [10], [8], [14], [15], [18], [4].)

The analysis of Galerkin methods for (CB) or (SCB) is of considerable interest due to the loss of optimal order of accuracy that emerges from the error estimates and is also supported by the numerical experiments. We shall investigate this phenomenon in detail in the paper but one may in general say that in the uniform spatial mesh case the (limited) loss of accuracy seems to be due to a combination of effects related to the

boundary conditions and the specific form of the ‘classical’ Boussinesq systems. (For example, no loss of accuracy occurs in the case of periodic boundary conditions on η and u for these systems, cf. [3], [9].) In the case of general (quasiuniform) mesh the (more severe) loss of accuracy seems to stem from the lack of cancellation effects that are present in the uniform mesh case, and from the hyperbolic character of the first p.d.e. of these systems. (The loss of optimal order of accuracy in standard Galerkin semidiscretizations of first-order hyperbolic equations manifested in various contexts is well known and was early observed e.g. by Dupont in [20].)

In section 2 we consider the standard Galerkin semidiscretizations of (CB) and (SCB) in the space of piecewise linear, continuous functions. In the case of a quasiuniform mesh, we prove that if the solution (η, u) of (CB) or (SCB) is sufficiently smooth and (η_h, u_h) is the semidiscrete approximation, then $\|\eta - \eta_h\| = O(h)$ and $\|u - u_h\|_1 = O(h)$. (Here $\|\cdot\|$, $\|\cdot\|_1$ denote, respectively, the L^2 and H^1 norms on $[0, 1]$.) In the case of uniform mesh, a suitable superaccuracy result for the error of the interpolant into the finite element subspace (a consequence of cancellations due to the uniform mesh) affords proving the improved estimates $\|\eta - \eta_h\| = O(h^{3/2})$ and $\|u - u_h\| = O(h^2)$. These rates of convergence are confirmed by the numerical experiments at the end of the section.

In section 3 we turn to the semidiscretization of (CB) or (SCB) using as finite element subspace the C^2 cubic splines. In the case of quasiuniform mesh we now get, as expected, $\|\eta - \eta_h\| = O(h^3)$, $\|u - u_h\|_1 = O(h^3)$, following the technique of the analogous proof in the previous section. Use of a uniform mesh and of relevant superconvergence results of Wahlbin, [30], allows proving a series of suitable superaccuracy estimates for the error of the interpolant and the error of the elliptic projection into the cubic spline subspaces. These results lead to error estimates such as $\|\eta - \eta_h\| = O(h^{3.5} \sqrt{\ln 1/h})$, $\|u - u_h\| = O(h^4 \sqrt{\ln 1/h})$, that are consistent with the rates of convergence of the errors obtained from numerical experiments.

In section 4 we turn to the analysis of fully discrete schemes. We consider only *explicit* time-stepping schemes in order to avoid the more costly implicit methods that require solving nonlinear systems of equations at every time step. Of course, with explicit methods there arises the issue of stability of the fully discrete schemes. We confine ourselves to a uniform mesh spatial discretization and consider three representative explicit Runge-Kutta schemes, namely, the Euler, the improved Euler, and the classical, four-stage, Runge-Kutta methods, whose orders of accuracy are 1, 2, and 4, respectively. We couple the Euler and improved Euler schemes with a piecewise linear spatial discretization and the fourth-order RK scheme with cubic splines. We show that the stability restrictions on the time step k needed by these schemes are of form $k = O(h^2)$, $k = O(h^{4/3})$, and $k \leq \lambda_0 h$ for a constant λ_0 sufficiently small, for the schemes with Euler, improved Euler, and fourth-order RK temporal discretizations, respectively. Under these restrictions, we prove optimal-order in time error estimates for the fully discrete schemes; the spatial rates of convergence are these of the semidiscrete approximations. The evidence of numerical experiments is consistent with the theoretical results.

In section 5 we analyze a nonstandard Galerkin semidiscrete approximation of (CB) and (SCB) that approximates η by piecewise linear continuous functions and u by piecewise quadratic C^1 functions defined on the same (uniform) mesh. (The method may be generalized using the analogous pairs of higher-order splines but here we confine ourselves to the low-order case.) A series of superaccuracy results for the error of the L^2 projection onto the space of piecewise linear continuous functions is developed, one of which makes use of a result of Demko, [17], on the exponential decay of the off-diagonal elements of the inverse of a certain tridiagonal matrix. With the aid of these results we show optimal-order error estimates e.g. of the type $\|\eta - \eta_h\| = O(h^2)$, $\|u - u_h\| = O(h^3)$. Finally, in section 6 we make some remarks on the application of standard Galerkin methods on some simple first-order hyperbolic problems. Specifically, superaccuracy tools that were developed in the case of uniform mesh in previous sections are combined with further similar estimates proved in this section to provide the basis for extending the optimal-order L^2 error estimate of Dupont, [20], for the periodic initial value problem, to the case of two types of initial-boundary value problems for first-order hyperbolics. The situation is contrasted with what happens in the case of nonuniform meshes by means of numerical examples.

In the paper we use the following notation: We let $C^k = C^k[0, 1]$, $k = 0, 1, 2, \dots$, denote the space of k times continuously differentiable functions on $[0, 1]$ and define $C_0^k = \{\phi \in C^k; \phi(0) = \phi(1) = 0\}$. We let for integer $k \geq 0$ H^k , $\|\cdot\|_k$ denote the usual, L^2 -based Sobolev space of classes of functions on $[0, 1]$ and its associated norm. (In the case $k = 1$ we use the equivalent norm defined by $\|v\|_1 = (\|v\|^2 + \frac{1}{3}\|v'\|^2)^{1/2}$.)

The inner product and norm on $L^2 = L^2(0, 1)$ we denote simply by $\|\cdot\|$, (\cdot, \cdot) , respectively. The norms on $L^\infty = L^\infty(0, 1)$ and on $W_\infty^k = W_\infty^k(0, 1)$ we denote by $\|\cdot\|_\infty$, $\|\cdot\|_{k,\infty}$, respectively. We let \mathbb{P}_r be the polynomials of degree $\leq r$, and by $\langle \cdot, \cdot \rangle$, $|\cdot|$, we denote the Euclidean inner product and norm on \mathbb{R}^N .

2. STANDARD GALERKIN SEMIDISCRETIZATION WITH PIECEWISE LINEAR, CONTINUOUS FUNCTIONS

2.1. Semidiscretization on a quasiuniform mesh. Let $0 = x_1 < x_2 < \dots < x_{N+1} = 1$ denote a quasiuniform partition of $[0, 1]$ with $h := \max_i(x_{i+1} - x_i)$, and let $S_h^2 := \{\phi \in C^0 : \phi|_{[x_j, x_{j+1}]} \in \mathbb{P}_1, 1 \leq j \leq N\}$, $S_{h,0}^2 = \{\phi \in S_h^2, \phi(0) = \phi(1) = 0\}$. Let $I_h, I_{h,0}$ denote the interpolation operators, with respect to the partition $\{x_j\}$, into the spaces $S_h^2, S_{h,0}^2$ respectively. Then, it is well known that there exists a constant C independent of h such that

$$\|w - I_h w\| + h\|(w - I_h w)'\| \leq Ch^k \|w^{(k)}\|, \quad (2.1)$$

for $w \in H^k$, $k = 1, 2$, and that a similar estimate holds for $I_{h,0} w$ if $w \in H^k \cap H_0^1$. (In the sequel, C will denote a generic constant, independent of h .) Let $a(\cdot, \cdot)$ denote the bilinear form

$$a(\psi, \chi) := (\psi, \chi) + \frac{1}{3}(\psi', \chi') \quad \forall \psi, \chi \in S_{h,0}^2,$$

and $R_h : H^1 \rightarrow S_{h,0}^2$ be the elliptic projection operator relative to $a(\cdot, \cdot)$, defined by

$$a(R_h w, \chi) = a(w, \chi) \quad \forall \chi \in S_{h,0}^2.$$

It follows by standard estimates that for $k = 0, 1$

$$\|R_h w - w\|_k \leq Ch^{2-k} \|w''\| \quad \text{if } w \in H^2 \cap H_0^1. \quad (2.2)$$

There also holds, [19], that

$$\|R_h w - w\|_\infty + h\|R_h w - w\|_{1,\infty} \leq Ch^2 \|w\|_{2,\infty}, \quad (2.3)$$

provided that $w \in W^{2,\infty} \cap H_0^1$.

As a consequence of the quasiuniformity of the mesh, the inverse inequalities

$$\|\chi\|_1 \leq Ch^{-1} \|\chi\|, \quad (2.4)$$

$$\|\chi\|_\infty \leq Ch^{-1/2} \|\chi\|, \quad (2.5)$$

hold for any $\chi \in S_h^2$ (or $\chi \in S_{h,0}^2$), and so does the estimate, [19],

$$\|Pv - v\|_\infty \leq Ch^2 \|v\|_{2,\infty}, \quad \text{for } v \in W^{2,\infty}, \quad (2.6)$$

where $P : L^2 \rightarrow S_h^2$ is the L^2 -projection operator onto S_h^2 .

The *standard Galerkin semidiscretization* on S_h^2 of (CB) is defined as follows: We seek $\eta_h : [0, T] \rightarrow S_h^2$, $u_h : [0, T] \rightarrow S_{h,0}^2$, such that for $t \in [0, T]$

$$\begin{aligned} (\eta_{ht}, \phi) + (u_{hx}, \phi) + ((\eta_h u_h)_x, \phi) &= 0 \quad \forall \phi \in S_h^2, \\ a(u_{ht}, \chi) + (\eta_{hx}, \chi) + (u_h u_{hx}, \chi) &= 0 \quad \forall \chi \in S_{h,0}^2, \end{aligned} \quad (2.7)$$

with initial conditions

$$\eta_h(0) = P\eta_0, \quad u_h(0) = R_h u_0. \quad (2.8)$$

Similarly, we define the analogous semidiscretization of (SCB), which is given for $0 \leq t \leq T$ by

$$\begin{aligned} (\eta_{ht}, \phi) + (u_{hx}, \phi) + \frac{1}{2}((\eta_h u_h)_x, \phi) &= 0 \quad \forall \phi \in S_h^2, \\ a(u_{ht}, \chi) + (\eta_{hx}, \chi) + \frac{3}{2}(u_h u_{hx}, \chi) + \frac{1}{2}(\eta_h \eta_{hx}, \chi) &= 0 \quad \forall \chi \in S_{h,0}^2, \end{aligned} \quad (2.9)$$

with

$$\eta_h(0) = P\eta_0, \quad u_h(0) = R_h u_0. \quad (2.10)$$

Upon choice of a basis for S_h^2 , it is seen that the semidiscrete problems (2.7)-(2.8) and (2.9)-(2.10) represent initial-value problems for systems of o.d.e's. Clearly, these systems have unique solutions at least locally in time. One conclusion of the next proposition is that they possess unique solutions up to $t = T$, where $[0, T]$ will denote henceforth the interval of existence of solutions of (CB) or (SCB).

Proposition 2.1. *Let h be sufficiently small. Suppose that the solutions of (CB), and (SCB), are such that $\eta \in C(0, T; W_\infty^2)$, $u \in C(0, T; W_\infty^2 \cap H_0^1)$. Then, the semidiscrete problems (2.7), (2.8) and (2.9), (2.10) have unique solutions (η_h, u_h) for $0 \leq t \leq T$ that satisfy*

$$\max_{0 \leq t \leq T} \|\eta(t) - \eta_h(t)\| \leq Ch, \quad (2.11)$$

$$\max_{0 \leq t \leq T} \|u(t) - u_h(t)\|_1 \leq Ch. \quad (2.12)$$

Proof. We first consider the case of the symmetric system (SCB). Putting $\phi = \eta_h$ and $\chi = u_h$ in (2.9) and adding the resulting equations, we obtain the discrete analog of (1.4), i.e. that

$$\|\eta_h(t)\|^2 + \|u_h(t)\|_1^2 = \|\eta_h(0)\|^2 + \|u_h(0)\|_1^2 \quad (2.13)$$

is valid in the temporal interval of existence of solutions of (2.9)-(2.10). By standard o.d.e. theory we conclude that the system (2.9)-(2.10) possesses a unique solution in any finite temporal interval $[0, T]$.

We now let $\rho := \eta - P\eta$, $\theta := P\eta - \eta_h$, $\sigma := u - R_h u$, $\xi := R_h u - u_h$. Using (SCB) and (2.9)-(2.10) we obtain for $0 \leq t \leq T$,

$$(\theta_t, \phi) + (\sigma_x + \xi_x, \phi) + \frac{1}{2}((\eta u - \eta_h u_h), \phi) = 0 \quad \forall \phi \in S_h^2, \quad (2.14)$$

$$a(\xi_t, \chi) + (\rho_x + \theta_x, \chi) + \frac{3}{2}(u u_x - u_h u_{hx}, \chi) + \frac{1}{2}(\eta \eta_x - \eta_h \eta_{hx}, \chi) = 0 \quad \forall \chi \in S_{h,0}^2. \quad (2.15)$$

Note that

$$\begin{aligned} \eta u - \eta_h u_h &= u(\rho + \theta) - (\rho + \theta)(\sigma + \xi) + \eta(\sigma + \xi), \\ u u_x - u_h u_{hx} &= (u\sigma)_x + (u\xi)_x - (\sigma\xi)_x - \sigma\sigma_x - \xi\xi_x, \\ \eta \eta_x - \eta_h \eta_{hx} &= (\eta\rho)_x + (\eta\theta)_x - (\rho\theta)_x - \rho\rho_x - \theta\theta_x. \end{aligned}$$

By continuity, in view of (2.10), we conclude that there exists a maximal temporal instance $t_h > 0$ such that $\|\theta(t)\|_\infty \leq 1$ for $t \leq t_h$. Suppose that $t_h < T$. Then, taking $\phi = \theta$ in (2.14) and using (2.1), (2.2), (2.3), (2.4), (2.6), and integrating by parts, we have for $0 \leq t \leq t_h$

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\theta\|^2 &= -(\sigma_x, \theta) - (\xi_x, \theta) - \frac{1}{2}[(\rho u)_x, \theta] + \frac{1}{2}(u_x \theta, \theta) - ((\rho\sigma)_x, \theta) \\ &\quad - ((\rho\xi)_x, \theta) - \frac{1}{2}(\sigma_x \theta, \theta) - \frac{1}{2}(\xi_x \theta, \theta) + ((\eta\sigma)_x, \theta) + ((\eta\xi)_x, \theta) \\ &\leq \|\sigma_x\| \|\theta\| + \|\xi_x\| \|\theta\| + \frac{1}{2} \|u\|_\infty \|\rho_x\| \|\theta\| + \frac{1}{2} \|u_x\|_\infty \|\rho\| \|\theta\| \\ &\quad + \frac{1}{4} \|u_x\|_\infty \|\theta\|^2 + \frac{1}{2} \|\rho_x\| \|\sigma\|_\infty \|\theta\| + \frac{1}{2} \|\rho\| \|\sigma_x\|_\infty \|\theta\| + \frac{1}{2} \|\rho\|_\infty \|\xi_x\| \|\theta\| \\ &\quad + C \|\rho_x\| \|\xi\|_1 \|\theta\| + \frac{1}{4} \|\sigma_x\|_\infty \|\theta\|^2 + \frac{1}{4} \|\xi_x\|_\infty \|\theta\| + \frac{1}{2} \|\eta\|_\infty \|\sigma_x\| \|\theta\| \\ &\quad + \frac{1}{2} \|\eta_x\|_\infty \|\sigma\| \|\theta\| + \frac{1}{2} \|\eta\|_\infty \|\xi_x\| \|\theta\| + \frac{1}{2} \|\eta_x\|_\infty \|\xi\| \|\theta\| \\ &\leq C(h + \|\xi\|_1 + \|\theta\|) \|\theta\|, \end{aligned} \quad (2.16)$$

where C is independent of t_h . In addition, putting $\chi = \xi$ in (2.15) we similarly obtain, for $0 \leq t \leq t_h$

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\xi\|_1^2 &= (\rho + \theta, \xi_x) + \frac{3}{2}[(u\sigma, \xi_x) + (u\xi, \xi_x) - (\sigma\xi, \xi_x) + (\sigma\sigma_x, \xi)] \\ &\quad + \frac{1}{2}[(\eta\rho, \xi_x) + (\eta\theta, \xi_x) - (\rho\theta, \xi_x) - \frac{1}{2}(\rho\xi_x, \rho) - (\theta\xi_x, \theta)] \\ &\leq \|\rho\| \|\xi_x\| + \|\theta\| \|\xi_x\| + \frac{3}{2} \|u\|_\infty \|\sigma\| \|\xi_x\| + \frac{3}{2} \|u\|_\infty \|\xi\| \|\xi_x\| \\ &\quad + \frac{3}{2} \|\sigma\|_\infty \|\xi\| \|\xi_x\| + \frac{3}{2} \|\sigma\|_\infty \|\sigma_x\| \|\xi\| + \frac{1}{2} \|\eta\|_\infty \|\rho\| \|\xi_x\| \\ &\quad + \frac{1}{2} \|\eta\|_\infty \|\theta\| \|\xi_x\| + \frac{1}{2} \|\rho\| \|\xi_x\| + C \|\rho\|_1 \|\rho\| \|\xi\|_1 + \frac{1}{2} \|\xi_x\| \|\theta\| \\ &\leq C(h^2 + \|\xi\|_1 + \|\theta\|) \|\xi\|_1. \end{aligned} \quad (2.17)$$

From (2.16) and (2.17) it is seen that for $0 \leq t \leq t_h$ there holds

$$\frac{d}{dt} (\|\theta\| + \|\xi\|_1) \leq C(h + \|\theta\| + \|\xi\|_1),$$

from which, by Gronwall's Lemma and (2.10), we conclude that

$$\|\theta(t)\| + \|\xi(t)\|_1 \leq Ch, \quad 0 \leq t \leq t_h, \quad (2.18)$$

where C is independent of t_h . Since by (2.5) $\|\theta\|_\infty \leq Ch^{-1/2}\|\theta\|$, if h is sufficiently small the maximality property of t_h is contradicted. Therefore we may take $t_h = T$, and (2.11) and (2.12) follow from (2.18), (2.1) and (2.2).

Consider now (CB) and its semidiscrete analog (2.7)-(2.8). The invariance property (2.13) no longer holds, and the i.v.p. (2.7)-(2.8) has a local unique solution. Using the same notation as in the case of (SCB), we consider the i.v.p. of finding $\theta(t) \in S_h^2$, $\xi(t) \in S_{h,0}^2$ for $t \geq 0$, such that

$$\begin{aligned} (\theta_t, \phi) + (\sigma_x + \xi_x, \phi) + ([u(\rho + \theta) - (\rho + \theta)(\sigma + \xi) + \eta(\sigma + \xi)], \phi) &= 0 \quad \forall \phi \in S_h^2, \\ a(\xi_t, \chi) + (\rho_x + \theta_x, \chi) - (u(\sigma + \xi) - \sigma\xi, \chi') - (\sigma\sigma_x + \xi\xi_x, \chi) &= 0 \quad \forall \chi \in S_{h,0}^2, \\ \theta(0) &= 0, \quad \xi(0) = 0. \end{aligned} \quad (2.19)$$

Obviously, (2.19) has a local unique solution. Let $t_h \in (0, T)$ be the maximal time instance for which this solution exists and satisfies $\|\theta(t)\|_\infty \leq 1$ for $0 \leq t \leq t_h$. Then, as in the case of (SCB), we obtain again that

$$\|\theta(t)\| + \|\xi(t)\|_1 \leq Ch, \quad 0 \leq t \leq t_h,$$

where C is independent of t_h . We conclude that we may take $t_h = T$. If $\eta_h = P\eta - \theta$, $u_h = R_h u - \xi$, it follows that (η_h, u_h) is a unique solution of (2.7)-(2.8) in $[0, T]$ and that it satisfies the estimates (2.11) and (2.12). \square

2.2. Uniform mesh. We now turn to the case of *uniform mesh*. For integer $N \geq 2$ we let $h = 1/N$ and $x_i = (i-1)h$, $i = 1, 2, \dots, N+1$. (We shall use the previously established notation for the finite-dimensional spaces S_h^2 , $S_{h,0}^2$, and their associated interpolation and projection operators.) We let $\{\phi_j\}_{j=1}^{N+1}$ denote the basis of S_h^2 satisfying $\phi_j(x_i) = \delta_{ij}$, $1 \leq i, j \leq N+1$. In the following lemma we collect results that will prove useful in the error estimates that follow.

Lemma 2.1. (i) Let $G_{ij} = (\phi_j, \phi_i)$, $1 \leq i, j \leq N+1$. Then there exist positive constants c_1 and c_2 such that

$$c_1 h |\gamma|^2 \leq \langle G\gamma, \gamma \rangle \leq c_2 h |\gamma|^2 \quad \forall \gamma \in \mathbb{R}^{N+1}.$$

(ii) Let $b \in \mathbb{R}^{N+1}$, $\gamma = G^{-1}b$, and $\psi = \sum_{j=1}^{N+1} \gamma_j \phi_j$. Then

$$\|\psi\| \leq (c_1 h)^{-1/2} |b|.$$

(iii) Let $w \in C^3$. Then, there exists a constant $C_1 = C_1(\|w^{(3)}\|_\infty)$ such that for any $\hat{x} \in [x_i, x_{i+1}]$

$$(w - I_h w)(x) = -\frac{1}{2} w''(\hat{x})(x - x_i)(x_{i+1} - x) + \tilde{g}(x), \quad x_i \leq x \leq x_{i+1},$$

where $\|\tilde{g}\|_\infty + h\|\tilde{g}'\|_\infty \leq C_1 h^3$.

Proof. The proofs of (i), (ii), (iii) are given in [20] in case the finite element subspace consists of continuous, piecewise linear, periodic functions on $[0, 1]$. It is straightforward to adapt them in the case of S_h^2 at hand. \square

The following superapproximation result, a consequence of cancellations due to the uniform mesh, is central for the sequel.

Lemma 2.2. Let $v \in C^3$ and $w \in C^1$. If $\varepsilon := v - I_h v$ and $\psi \in S_h^2$ is such that

$$(\psi, \phi) = ((w\varepsilon)', \phi) \quad \forall \phi \in S_h^2,$$

then $\|\psi\| \leq Ch^{3/2}$. If in addition $w(0) = w(1) = 0$, then $\|\psi\| \leq Ch^2$.

Proof. Let $b_i := ((w\varepsilon)', \phi_i) = -(w\varepsilon, \phi_i')$, $1 \leq i \leq N+1$. Clearly, $|b_1| = O(h^2)$ and $|b_{N+1}| = O(h^2)$. Let $2 \leq i \leq N$. Then, from Lemma 2.1(iii) we have

$$\begin{aligned} b_i &= \frac{v''(x_i)}{2h} \int_{x_{i-1}}^{x_i} w(x)(x - x_{i-1})(x_i - x)dx - \frac{v''(x_i)}{2h} \int_{x_i}^{x_{i+1}} w(x)(x - x_i)(x_{i+1} - x)dx + O(h^3) \\ &= \frac{-v''(x_i)}{2h} \int_{x_i}^{x_{i+1}} [w(x) - w(x-h)](x - x_i)(x_{i+1} - x)dx + O(h^3) \\ &= \frac{-v''(x_i)}{2} \int_{x_i}^{x_{i+1}} w'(\xi_x)(x - x_i)(x_{i+1} - x)dx + O(h^3). \end{aligned}$$

Therefore, $b_i = O(h^3)$, $2 \leq i \leq N$; consequently $|b| = O(h^2)$ and $\|\psi\| = O(h^{3/2})$ by Lemma 2.1(ii). If $w(0) = 0$, Lemma 2.1(iii) gives

$$\begin{aligned} b_1 &= -(w\varepsilon, \phi'_1) = \frac{-v''(0)}{2h} \int_0^h w(x)x(h-x)dx + O(h^3) \\ &= \frac{-v''(0)}{2} \int_0^h w'(\xi_x)x^2(h-x)dx + O(h^3) = O(h^3). \end{aligned}$$

Similarly, if $w(1) = 0$, then $b_{N+1} = O(h^3)$. Hence, $|b| = O(h^{5/2})$, giving $\|\psi\| = O(h^2)$ by Lemma 2.1(ii). \square

In the uniform mesh case at hand we consider again the semidiscretizations (2.7) and (2.9) of (CB) and (SCB) respectively, with the initial conditions for both systems now given by the interpolants of η_0, u_0 :

$$\eta_h(0) = I_h\eta_0, \quad u_h(0) = I_{h,0}u_0. \quad (2.20)$$

The main result of this paragraph is

Theorem 2.1. *Let $h = 1/N$ be sufficiently small. Suppose that the solutions of (CB) and (SCB) are such that $\eta \in C(0, T; C^3)$, $\eta_t \in C(0, T; C^2)$, $u, u_t \in C(0, T; C_0^3)$. Then, the semidiscrete problems (2.7), (2.20) and (2.9), (2.20) have unique solutions (η_h, u_h) for $0 \leq t \leq T$ that satisfy*

$$\begin{aligned} (i) \quad & \max_{0 \leq t \leq T} \|\eta(t) - \eta_h(t)\| \leq Ch^{3/2}, \quad \max_{0 \leq t \leq T} \|u(t) - u_h(t)\|_1 \leq Ch, \\ (ii) \quad & \max_{0 \leq t \leq T} \|u(t) - u_h(t)\| \leq Ch^2, \quad \max_{0 \leq t \leq T} \|u_t(t) - u_{ht}(t)\| \leq Ch^2. \end{aligned}$$

Proof. We give the proof in detail in the case of (SCB), where existence of solutions of the i.v.p. (2.9), (2.20) for $0 \leq t \leq T$ follows from (2.13). The proof in the case of (CB) follows from an argument analogous to that given in the proof of Proposition 2.1 and will be omitted. Let

$$\rho := \eta - I_h\eta, \quad \theta := I_h\eta - \eta_h, \quad \sigma := u - I_{h,0}u, \quad \xi := I_{h,0}u - u_h.$$

Note that

$$\eta u - \eta_h u_h = \eta\sigma + u\theta - \theta\xi + F,$$

where

$$F := \eta\xi + u\rho - \rho\sigma - \rho\xi - \theta\sigma.$$

In addition

$$\eta\eta_x - \eta_h\eta_{hx} = -\theta\theta_x + (\eta\theta)_x + G_x,$$

where

$$G := \eta\rho - \rho\theta - \frac{1}{2}\rho^2,$$

and

$$uu_x - u_h u_{hx} = H_x,$$

where

$$H := u\sigma + u\xi - \sigma\xi - \frac{1}{2}\sigma^2 - \frac{1}{2}\xi^2.$$

Then, from (SCB) and (2.9), (2.20) it follows for $0 \leq t \leq T$ that

$$\begin{aligned} (\theta_t, \phi) + (\xi_x, \phi) + \left(\left((1 + \frac{1}{2}\eta)\sigma \right)_x, \phi \right) + \frac{1}{2}((u\theta)_x, \phi) - \frac{1}{2}((\theta\xi)_x, \phi) \\ + \frac{1}{2}(F_x, \phi) = -(\rho_t, \phi) \quad \forall \phi \in S_h^2, \end{aligned} \quad (2.21)$$

$$\begin{aligned} a(\xi_t, \chi) + (\theta_x, \chi) + (\rho_x, \chi) - \frac{1}{2}(\theta\theta_x, \chi) + \frac{1}{2}((\eta\theta)_x, \chi) + \frac{1}{2}(G_x, \chi) \\ + \frac{3}{2}(H_x, \chi) = -(\sigma_t, \chi) \quad \forall \chi \in S_{h,0}^2, \end{aligned} \quad (2.22)$$

with

$$\theta(0) = 0, \quad \xi(0) = 0. \quad (2.23)$$

(In the right-hand side of (2.22) we used the fact that for $\chi \in S_{h,0}^2$ $a(\sigma_t, \chi) = (\sigma_t, \chi)$, since $(v' - (I_{h,0}v)', \chi') = 0$ for $v \in H_0^1$.) In order to show the estimates in (i), we put $\phi = \theta$ and $\chi = \xi$ in (2.21) and (2.22), integrate by parts, and add the resulting equations to get for $0 \leq t \leq T$

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (\|\theta\|^2 + \|\xi\|_1^2) + ((1 + \frac{1}{2}\eta)\sigma)_x, \theta + \frac{1}{2}((u\theta)_x, \theta) + \frac{1}{2}(F_x, \theta) \\ + (\rho_x, \xi) + \frac{1}{2}((\eta\theta)_x, \xi) + \frac{1}{2}(G_x, \xi) + \frac{3}{2}(H_x, \xi) = -(\rho_t, \theta) - (\sigma_t, \xi). \end{aligned} \quad (2.24)$$

Using now the approximation properties of S_h^2 and $S_{h,0}^2$, integration by parts, and Lemmas 2.1 and 2.2 we see that

$$\begin{aligned} |((1 + \frac{1}{2}\eta)\sigma)_x, \theta| &= |P[(1 + \frac{1}{2}\eta)\sigma]_x, \theta| \leq Ch^{3/2}\|\theta\|, \\ |(u\theta)_x, \theta| &= \frac{1}{2}|(u_x\theta, \theta)| \leq C\|\theta\|^2, \\ |(F_x, \theta)| &\leq |((\eta\xi)_x, \theta)| + |((u\rho)_x, \theta)| + |((\rho\sigma)_x, \theta)| + |((\rho\xi)_x, \theta)| + \frac{1}{2}|(\sigma_x\theta, \theta)| \\ &\leq C(\|\xi\|_1\|\theta\| + h^2\|\theta\| + h^3\|\theta\| + h\|\xi\|_1\|\theta\| + h\|\theta\|^2) \\ &\leq C(\|\xi\|_1\|\theta\| + h^2\|\theta\| + h\|\theta\|^2), \\ |(\rho_x, \xi)| &\leq Ch^2\|\xi\|_1, \\ |((\eta\theta)_x, \xi)| &\leq C\|\xi\|_1\|\theta\|, \\ |(G_x, \xi)| &\leq |(\eta\rho, \xi_x)| + |(\rho\theta, \xi_x)| + \frac{1}{2}|(\rho^2, \xi_x)| \\ &\leq C(h^2\|\xi\|_1 + h^2\|\theta\|\|\xi\|_1 + h^4\|\xi\|_1) \\ &\leq C(h^2\|\xi\|_1 + h^2\|\theta\|\|\xi\|_1), \\ |(H_x, \xi)| &\leq |(u\sigma, \xi_x)| + \frac{1}{2}|(u_x\xi, \xi)| + |(\sigma\xi, \xi_x)| + \frac{1}{2}|(\sigma^2, \xi_x)| \\ &\leq C(h^2\|\xi\|_1 + \|\xi\|^2 + h^2\|\xi\|_1^2 + h^4\|\xi\|_1) \\ &\leq C(h^2\|\xi\|_1 + \|\xi\|_1^2), \\ |(\rho_t, \theta)| + |(\sigma_t, \xi)| &\leq C(h^2\|\theta\| + h^2\|\xi\|). \end{aligned}$$

Hence, from (2.24) we conclude that for $0 \leq t \leq T$

$$\begin{aligned} \frac{d}{dt} (\|\theta\|^2 + \|\xi\|_1^2) &\leq C(h^{3/2}\|\theta\| + h^2\|\xi\|_1) \\ &\quad + C(\|\theta\|^2 + \|\xi\|_1^2) \leq C[h^3 + (\|\theta\|^2 + \|\xi\|_1^2)]. \end{aligned}$$

Hence, by Gronwall's Lemma we obtain

$$\|\theta(t)\|^2 + \|\xi(t)\|_1^2 \leq C[h^3 + \|\theta(0)\|^2 + \|\xi(0)\|_1^2],$$

from which, in view of (2.20), we get

$$\|\theta\| + \|\xi\|_1 \leq Ch^{3/2}, \quad 0 \leq t \leq T, \quad (2.25)$$

and the estimates in (i) follow. In addition, from (2.25) and the approximation properties of S_h^2 and $S_{h,0}^2$ one may easily derive the following L^2 estimates of F , G , and H , that we note for further reference

$$\begin{aligned} \|F\| &\leq C(\|\xi\| + h^2), \\ \|G\| &\leq Ch^2, \\ \|H\| &\leq C(\|\xi\| + h^2). \end{aligned} \quad (2.26)$$

We proceed now to prove the optimal-order error estimates in (ii). Equation (2.22) may be written in the form

$$\xi_t = R_h v, \quad (2.27)$$

where v is the solution of the problem

$$\begin{aligned} v - \frac{1}{3}v'' &= -(\theta + \rho)_x - \frac{1}{2}(\eta\theta)_x - \frac{1}{2}(G - \frac{1}{2}\theta^2 + 3H)_x - \sigma_t, \quad x \in [0, 1], \\ v(0) &= v(1) = 0. \end{aligned} \quad (2.28)$$

Considering the weak form of (2.28) in H_0^1 , and using integration by parts in the right-hand side, (2.25) and (2.26) we see that

$$\|v\|_1 \leq Ch^{3/2}. \quad (2.29)$$

In order to derive a bound for $\|v\|$, let $\zeta \in L^2$ and V be the solution of the problem

$$\begin{aligned} V - \frac{1}{3}V'' &= \zeta, \quad x \in [0, 1], \\ V(0) &= V(1) = 0. \end{aligned} \quad (2.30)$$

Then, by (2.28)

$$(v, \zeta) = a(v, V) = (\theta + \rho, V') + \frac{1}{2}(\eta\theta, V') + \frac{1}{2}(G - \frac{1}{2}\theta^2 + 3H, V') - (\sigma_t, V). \quad (2.31)$$

From (2.21) with $\phi = 1$ we see that $(\theta_t + \rho_t, 1) = 0$, $0 \leq t \leq T$. Hence,

$$\int_0^1 (\theta + \rho) dx = \int_0^1 \rho(x, 0) dx =: J = \text{const}.$$

Therefore, if for $(x, t) \in [0, 1] \times [0, T]$

$$\gamma(x, t) := \int_0^x (\theta(s, t) + \rho(s, t)) ds - xJ, \quad (2.32)$$

it follows that $\gamma \in H_0^1$ and $\gamma_x = \theta + \rho - J$. Hence, (2.31) yields

$$\begin{aligned} (v, \zeta) &= (\gamma_x, V') + \frac{1}{2}(\eta\gamma_x, V') + \frac{1}{2}(G - \frac{1}{2}\theta^2 + 3H - \eta\rho + \eta J, V') - (\sigma_t, V) \\ &= -(\gamma, V'') - \frac{1}{2}(\gamma, \eta V'') + \frac{1}{2}(G - \frac{1}{2}\theta^2 + 3H - \eta\rho + \eta J, V') - (\sigma_t, V). \end{aligned}$$

Now, using (2.25), (2.26), the approximation and inverse properties of S_h^2 , $S_{h,0}^2$, and elliptic regularity in (2.30) we obtain

$$\begin{aligned} |(v, \zeta)| &\leq \|\gamma\| \|V''\| + C\|\gamma\| (\|V''\| + \|V'\|) + C(h^2 + \|\xi\|) \|V'\| + Ch^2 \|V\| \\ &\leq C(h^2 + \|\gamma\| + \|\xi\|) \|\zeta\|, \end{aligned}$$

and conclude that

$$\|v\| \leq C(h^2 + \|\gamma\| + \|\xi\|). \quad (2.33)$$

Let now W the solution of the problem

$$\begin{aligned} W - \frac{1}{3}W'' &= \xi, \quad x \in [0, 1], \\ W(0) &= W(1) = 0. \end{aligned} \quad (2.34)$$

Using (2.27), (2.15), elliptic regularity in (2.34), and the estimates (2.29) and (2.33) gives

$$\begin{aligned} (\xi, \xi_t) &= a(W, \xi_t) = a(W, R_h v) = a(R_h W, R_h v) \\ &= a(R_h v, R_h W) = a(v, R_h W) = a(v, R_h W - W) + a(W, v) \\ &= a(v, R_h W - W) + (\xi, v) \leq Ch\|\xi\| \|v\|_1 + \|\xi\| \|v\| \\ &\leq C(h^2 + \|\gamma\| + \|\xi\|) \|\xi\|, \end{aligned}$$

from which there follows that

$$\frac{1}{2} \frac{d}{dt} \|\xi\|^2 \leq C(h^4 + \|\gamma\|^2 + \|\xi\|^2), \quad 0 \leq t \leq T. \quad (2.35)$$

In order to obtain the required optimal-order estimate for $\|\xi\|$ from (2.35) we need a similar estimate for a suitable approximation of γ . For this purpose, observe that (2.21) yields, for $0 \leq t \leq T$ and $\phi \in S_h^2$,

$$(\gamma_{xt}, \phi) + \frac{1}{2}((u\theta)_x, \phi) = -(w_x, \phi), \quad (2.36)$$

where $w = \xi + (1 + \frac{1}{2}\eta)\sigma - \frac{1}{2}\theta\xi + \frac{1}{2}F$; note that $w \in H_0^1$ and that (2.25) and (2.26) give $\|w\| \leq C(\|\xi\| + h^2)$. Using integration by parts and the definition of γ in (2.36) yields for $0 \leq t \leq T$

$$(\gamma_t, \phi') + \frac{1}{2}(u\gamma_x, \phi') = -(w - \frac{1}{2}u\rho + \frac{1}{2}uJ, \phi') \quad \forall \phi \in S_h^2. \quad (2.37)$$

Consider now the space S_h^{-1} of discontinuous, piecewise constant functions relative to the partition $\{x_j\}$. Given any $\psi \in S_h^{-1}$ consider in (2.37) $\phi \in S_h^2$ such that $\phi' = \psi$. Hence we have for $0 \leq t \leq T$

$$(\gamma_t, \psi) + \frac{1}{2}(u\gamma_x, \psi) = -(K, \psi) \quad \forall \psi \in S_h^{-1},$$

where $K := w - \frac{1}{2}u\rho + \frac{1}{2}uJ$ satisfies $\|K\| \leq C(\|\xi\| + h^2)$. Taking now $\psi = P_0\gamma$ in the above, where P_0 is the L^2 -projection operator onto S_h^{-1} , yields for $0 \leq t \leq T$

$$\frac{1}{2} \frac{d}{dt} \|P_0\gamma\|^2 + \frac{1}{2}(u\gamma_x, P_0\gamma) = -(K, P_0\gamma). \quad (2.38)$$

Since $\|P_0\gamma - \gamma\| \leq Ch\|\gamma\|_1$, cf. e.g. (16.24) of [29], and $\|\gamma\|_1 \leq Ch^{3/2}$ by (2.25), we have

$$\begin{aligned} |(u\gamma_x, P_0\gamma)| &= |(u\gamma_x, P_0\gamma - \gamma) - \frac{1}{2}(u_x\gamma, \gamma)| \leq C(h\|\gamma\|_1^2 + \|\gamma\|^2) \\ &\leq C(h\|\gamma\|_1^2 + \|P_0\gamma - \gamma\|^2 + \|P_0\gamma\|^2) \\ &\leq C(h^4 + \|P_0\gamma\|^2). \end{aligned}$$

Hence, (2.38) yields, for $0 \leq t \leq T$, that

$$\frac{1}{2} \frac{d}{dt} \|P_0\gamma\|^2 \leq C(h^4 + \|P_0\gamma\|^2 + \|\xi\|^2). \quad (2.39)$$

Now, from (2.35), since $\|\gamma\| \leq \|P_0\gamma - \gamma\| + \|P_0\gamma\| \leq Ch^{5/2} + \|P_0\gamma\|$, we get for $0 \leq t \leq T$

$$\frac{1}{2} \frac{d}{dt} \|\xi\|^2 \leq C(h^4 + \|P_0\gamma\|^2 + \|\xi\|^2). \quad (2.40)$$

Adding (2.39) and (2.40) we finally obtain by Gronwall's Lemma and (2.23) that

$$\|P_0\gamma\|^2 + \|\xi\|^2 \leq Ch^4, \quad 0 \leq t \leq T. \quad (2.41)$$

Therefore, the first inequality of the conclusion (ii) of the theorem holds; in addition, by similar estimates with the ones used above, we also obtain

$$\|\gamma\| \leq Ch^2, \quad 0 \leq t \leq T. \quad (2.42)$$

Finally, we prove the second estimate of (ii). Let Z be the solution of the problem

$$\begin{aligned} Z - \frac{1}{3}Z'' &= \xi_t, \quad x \in [0, 1], \\ Z(0) &= Z(1) = 0. \end{aligned}$$

Then, by (2.27)

$$\|\xi_t\|^2 = a(Z, \xi_t) = a(Z, R_h v) = a(R_h Z, v) = a(R_h Z - Z, v) + (\xi_t, v).$$

Hence, elliptic regularity and (2.29), (2.33), (2.41), (2.42) give

$$\|\xi_t\|^2 \leq Ch\|Z\|_2\|v\|_1 + \|\xi_t\|\|v\| \leq Ch^2\|\xi_t\|,$$

i.e. $\|\xi_t\| \leq Ch^2$, and the second estimate of (ii) follows. \square

Remark 2.1. It is not hard to see that the conclusions of Theorem 2.1 hold if we take $\eta_h(0)$ as any approximation of η_0 in S_h^2 with optimal-order L^2 rate of convergence. However, $u_h(0)$ has to be taken as $I_{h,0}u_0$ or $R_h u_0$.

Remark 2.2. The superaccuracy estimate $\|\xi\|_1 = O(h^{3/2})$ of (2.25) combined with (2.3) and Sobolev's inequality yields the L^∞ estimate $\|u - u_h\|_\infty = O(h^{3/2})$ for u .

Remark 2.3. In the case of a *quasiuniform* mesh with $h = \max_i(x_{i+1} - x_i)$, one may easily check that the analog of Lemma 2.1 still holds; however in the proof of Lemma 2.2 there is no cancellation anymore from adjacent intervals and the conclusion is just that $\|\psi\| \leq Ch$. As a consequence, the techniques of the proof of Theorem 2.1 now yield that $\|\eta - \eta_h\| = O(h)$ and $\|u - u_h\|_1 = O(h)$, and, instead of the optimal-order estimates in (ii), that $\|u - u_h\| = O(h^{3/2})$ and $\|u_t - u_{ht}\| = O(h^{3/2})$. However, for the *linearized* problem

$$\begin{aligned} \eta_t + u_x &= 0, \\ u_t + \eta_x - \frac{1}{3}u_{xxt} &= 0, \quad (x, t) \in [0, 1] \times [0, T], \\ u(0, t) &= 0, \quad u(1, t) = 0, \quad t \in [0, T], \\ \eta(x, 0) &= \eta_0(x), \quad u(x, 0) = u_0(x), \quad x \in [0, 1], \end{aligned} \quad (2.43)$$

the last two estimates may be improved to yield optimal order, i.e. to give $\|u - u_h\| = O(h^2)$, $\|u_t - u_{ht}\| = O(h^2)$. Our numerical experiments suggest that $\|u - u_h\| = O(h^2)$ even in the nonlinear case, but we have not been able to prove this thusfar.

2.3. Numerical experiments. We first consider the case of approximations on a uniform mesh with $h = 1/N$ on $[0, 1]$. Table 2.1 shows the errors and the associated rates of convergence, in the L^2 -, L^∞ - and

L^2 -errors

N	η	order	u	order
40	0.1894(-1)		0.1749(-3)	
80	0.6849(-2)	1.467	0.4259(-4)	2.038
120	0.3761(-2)	1.478	0.1877(-4)	2.021
160	0.2454(-2)	1.484	0.1051(-4)	2.015
200	0.1761(-2)	1.487	0.6710(-5)	2.011
240	0.1342(-2)	1.490	0.4652(-5)	2.009
280	0.1066(-2)	1.491	0.3413(-5)	2.008
320	0.8738(-3)	1.492	0.2611(-5)	2.007
360	0.7328(-3)	1.493	0.2062(-5)	2.006
400	0.6261(-3)	1.494	0.1669(-5)	2.005
440	0.5430(-3)	1.495	0.1379(-5)	2.005
480	0.4767(-3)	1.495	0.1158(-5)	2.004
520	0.4230(-3)	1.496	0.9864(-6)	2.004

L^∞ -errors

N	η	order	u	order
40	0.1126		0.5916(-3)	
80	0.5617(-1)	1.004	0.1509(-3)	1.971
120	0.3741(-1)	1.002	0.6755(-4)	1.983
160	0.2804(-1)	1.002	0.3813(-4)	1.988
200	0.2243(-1)	1.001	0.2445(-4)	1.991
240	0.1868(-1)	1.001	0.1701(-4)	1.992
280	0.1601(-1)	1.001	0.1251(-4)	1.994
320	0.1401(-1)	1.001	0.9582(-5)	1.994
360	0.1245(-1)	1.001	0.7575(-5)	1.995
400	0.1121(-1)	1.001	0.6139(-5)	1.996
440	0.1019(-1)	1.001	0.5075(-5)	1.996
480	0.9337(-2)	1.001	0.4266(-5)	1.996
520	0.8618(-2)	1.001	0.3636(-5)	1.997

H^1 -errors

N	η	order	u	order
40	0.2440(+1)		0.2449(-1)	
80	0.1776(+1)	0.459	0.1192(-1)	1.039
120	0.1466(+1)	0.473	0.7875(-2)	1.022
160	0.1277(+1)	0.480	0.5880(-2)	1.016
200	0.1146(+1)	0.484	0.4691(-2)	1.012
240	0.1049(+1)	0.487	0.3902(-2)	1.010
280	0.9725	0.489	0.3341(-2)	1.008
320	0.9109	0.490	0.2920(-2)	1.007
360	0.8596	0.492	0.2594(-2)	1.006
400	0.8161	0.493	0.2333(-2)	1.006
440	0.7787	0.493	0.2120(-2)	1.005
480	0.7459	0.494	0.1942(-2)	1.005
520	0.7170	0.494	0.1792(-2)	1.004

TABLE 2.1. Errors and orders of convergence. (CB) system, standard Galerkin semidiscretization with piecewise linear, continuous functions on a uniform mesh.

H^1 -norms at $T = 1$, of the standard Galerkin approximation (using piecewise linear, continuous functions) to the (CB) system with suitable right-hand so that its exact solution is given by $\eta = \exp(2t)(\cos(\pi x) + x + 2)$, $u = \exp(-xt)x \sin(\pi x)$. The system was integrated up to $T = 1$ with the classical, four-stage, fourth-order explicit Runge-Kutta method (see Section 4) using a time step $k = h/10$. We checked that the temporal error of the discretization was very small compared to the spatial error, so that the errors and rates of convergence shown are essentially those of the semidiscrete problem (2.7), (2.20). The table suggests that the L^2 rates of convergence of η and u approach 3/2 and 2, respectively, thus confirming the relevant estimates of Theorem 2.1. It also suggests that $\|u - u_h\|_1 = O(h)$ (confirming the result of Theorem 2.1), and $\|\eta - \eta_h\|_1 = O(h^{1/2})$, $\|\eta - \eta_h\|_\infty = O(h)$, $\|u - u_h\|_\infty = O(h^2)$. Table 2.2 shows the analogous errors and rates in the case of a nonhomogeneous (SCB) system with the same exact solution $\eta = \exp(2t)(\cos(\pi x) + x + 2)$, $u = \exp(-xt)x \sin(\pi x)$, integrated on a uniform spatial mesh up to $T = 1$ with the same time stepping procedure. The convergence rates are essentially the same. Table 2.3(a) shows some results from a computation of the exact solution $\eta = \exp(2t)(\cos(\pi x) + x + 2)$, $u = \exp(xt)(\sin(\pi x) + x^3 - x^2)$ of a suitably nonhomogeneous version of the (CB) system with the standard Galerkin method with piecewise linear, continuous functions on the quasiuniform mesh on $[0, 1]$ given by $h_{2i-1} = 1.2\Delta x$, $h_{2i} = 0.8\Delta x$, $1 \leq i \leq N/2$, where $h_i = x_{i+1} - x_i$ and $\Delta x = 1/N$. Again the fully discrete problem was solved by the fourth-order accurate four-stage classical explicit Runge-kutta scheme with timestep $k = \Delta x/10$. We integrated the system up to $T = 0.4$ starting with the L^2 -projections of η_0 and u_0 on the finite element subspaces. The temporal error was much smaller than the spatial error. Table 2.3(a) shows the L^2 -errors for η and u and the associated rates of convergence

L^2 -errors

N	η	order	u	order
40	0.7423(-2)		0.3613(-3)	
80	0.2678(-2)	1.471	0.8849(-4)	2.030
120	0.1469(-2)	1.481	0.3907(-4)	2.016
160	0.9579(-3)	1.486	0.2190(-4)	2.011
200	0.6871(-3)	1.489	0.1399(-4)	2.009
240	0.5236(-3)	1.491	0.9703(-5)	2.007
280	0.4160(-3)	1.492	0.7122(-5)	2.006
320	0.3408(-3)	1.493	0.5449(-5)	2.005
360	0.2858(-3)	1.494	0.4303(-5)	2.005
400	0.2441(-3)	1.495	0.3484(-5)	2.004
440	0.2117(-3)	1.495	0.2878(-5)	2.004
480	0.1859(-3)	1.496	0.2418(-5)	2.003
520	0.1649(-3)	1.496	0.2060(-5)	2.003

 L^∞ -errors

N	η	order	u	order
40	0.3491(-1)		0.9655(-3)	
80	0.1704(-1)	1.034	0.2537(-3)	1.928
120	0.1125(-1)	1.024	0.1140(-3)	1.972
160	0.8394(-2)	1.018	0.6442(-4)	1.985
200	0.6694(-2)	1.014	0.4132(-4)	1.991
240	0.5567(-2)	1.012	0.2873(-4)	1.994
280	0.4764(-2)	1.010	0.2112(-4)	1.995
320	0.4164(-2)	1.008	0.1618(-4)	1.996
360	0.3698(-2)	1.008	0.1279(-4)	1.997
400	0.3326(-2)	1.007	0.1036(-4)	1.998
440	0.3021(-2)	1.006	0.8563(-5)	1.998
480	0.2768(-2)	1.006	0.7197(-5)	1.998
520	0.2554(-2)	1.005	0.6133(-5)	1.999

 H^1 -errors

N	η	order	u	order
40	0.1049(+1)		0.2670(-1)	
80	0.7383	0.506	0.1249(-1)	1.096
120	0.6026	0.501	0.8131(-2)	1.059
160	0.5219	0.500	0.6024(-2)	1.042
200	0.4668	0.500	0.4784(-2)	1.033
240	0.4262	0.499	0.3967(-2)	1.027
280	0.3946	0.499	0.3388(-2)	1.023
320	0.3692	0.499	0.2956(-2)	1.020
360	0.3481	0.500	0.2622(-2)	1.018
400	0.3302	0.500	0.2356(-2)	1.016
440	0.3149	0.500	0.2139(-2)	1.015
480	0.3015	0.500	0.1959(-2)	1.013

TABLE 2.2. Errors and orders of convergence. (SCB) system, standard Galerkin semidiscretization with piecewise linear, continuous functions on a uniform mesh.

N	η	order	u	order
80	0.1277(-1)		0.7432(-4)	
160	0.6383(-2)	1.000	0.1858(-4)	2.000
240	0.4258(-2)	0.999	0.8259(-5)	2.000
320	0.3194(-2)	0.999	0.4646(-5)	2.000
400	0.2556(-2)	0.999	0.2973(-5)	2.000
480	0.2131(-2)	0.999	0.2065(-5)	2.000
560	0.1826(-2)	0.999	0.1517(-5)	2.000
640	0.1598(-2)	0.999	0.1161(-5)	2.000
720	0.1421(-2)	0.999	0.9177(-5)	2.000

(a)

N	η	order	u	order
40	0.5852(-1)		0.1693(-2)	
80	0.2933(-1)	0.997	0.4271(-3)	1.987
120	0.1942(-1)	1.017	0.1899(-3)	2.000
160	0.1449(-1)	1.019	0.1068(-3)	2.000
200	0.1155(-1)	1.016	0.6834(-4)	2.001
240	0.9600(-2)	1.014	0.4745(-4)	2.001
280	0.8213(-2)	1.012	0.3486(-4)	2.001
320	0.7176(-2)	1.011	0.2669(-4)	2.001
360	0.6371(-2)	1.010	0.2109(-4)	2.000
400	0.5729(-2)	1.009	0.1708(-4)	2.001

(b)

TABLE 2.3. L^2 -errors and orders of convergence. (CB) system, standard Galerkin semidiscretization with piecewise linear, continuous functions on a quasiuniform mesh with $\frac{\max h_i}{\min h_i} = 1.5$ (a), $\frac{\max h_i}{\min h_i} = 150$ (b)

at $T = 0.4$. The data strongly suggest that $\|\eta - \eta_h\| = O(h)$ and $\|u - u_h\| = O(h^2)$, thus confirming the relevant theoretical result for η (cf. Proposition 2.1 and Remark 2.2), and supporting the conjecture that the L^2 rate of convergence for u is actually equal to 2 even in the case of the nonlinear problem; recall that the technique of proof of Theorem 2.1 in the case of a quasiuniform mesh gives a pessimistic bound of $O(h^{3/2})$ for $\|u - u_h\|$, cf. Remark 2.2. These results are confirmed by the rates shown in Table 2.3(b), which was obtained by integrating the same problem with the same method on the quasiuniform mesh on $[0, 1]$ with $h_{10i-9} = 0.02\Delta x$, $h_{10i-8} = 0.05\Delta x$, $h_{10i-7} = 0.08\Delta x$, $h_{10i-6} = 0.35\Delta x$, $h_{10i-5} = 0.5\Delta x$, $h_{10i-4} = h_{10i-3} = \Delta x$, $h_{10i-2} = h_{10i-1} = 2\Delta x$, $h_{10i} = 3\Delta x$, $1 \leq i \leq N/10$, and $k = \Delta x/10$.

3. STANDARD GALERKIN SEMIDISCRETIZATION WITH CUBIC SPLINES

3.1. Semidiscretization on a quasiuniform mesh. Let $0 = x_1 < x_2 < \dots < x_{N+1} = 1$ denote a quasiuniform partition of $[0, 1]$ with $h := \max_i (x_{i+1} - x_i)$ and let

$$S_h^4 := \{\phi \in C^2 : \phi|_{[x_j, x_{j+1}]} \in \mathbb{P}_3, 1 \leq j \leq N\}, \quad S_{h,0}^4 = \{\phi \in S_h^4 : \phi(0) = \phi(1) = 0\},$$

be the space of (the C^2) cubic splines on $[0, 1]$ relative to the partition $\{x_j\}$, and the space of cubic splines that vanish at $x = 0$ and at $x = 1$. In this section we shall denote by $I_h : C^1 \rightarrow S_h^4$ the interpolation operator, with the properties that for any $v \in C^1$, $(I_h v)(x_i) = v(x_i)$, $1 \leq i \leq N+1$, $(I_h v)'(x_k) = v'(x_k)$, $k = 1, N+1$, and let $I_{h,0} : C_0^1 \rightarrow S_{h,0}^4$ be the analogous interpolant onto $S_{h,0}^4$. It is well known that

$$\sum_{j=0}^2 h^j \|w - I_h w\|_j \leq Ch^k \|w^{(k)}\| \quad (3.1)$$

holds for any $w \in H^k$ for $k = 2, 3, 4$ and that a similar estimate holds for $I_{h,0} w$ if $w \in H^k \cap H_0^1$. More generally, [28], if $1 \leq k \leq 4$ and $0 \leq j < k$ we have that

$$\min_{\chi \in S_h^4} \|(w - \chi)^{(j)}\| \leq Ch^{k-j} \|w^{(k)}\| \quad \text{if } w \in H^k$$

and

$$\min_{\chi \in S_h^4} \|(w - \chi)^{(j)}\|_\infty \leq Ch^{k-j} \|w^{(k)}\|_\infty \quad \text{if } w \in C^k,$$

and that a similar estimate holds in $S_{h,0}^4$ for w in H^k or in C^k that also vanishes at $x = 0$ and $x = 1$. Let $a(\cdot, \cdot)$ denote again the bilinear form

$$a(\psi, \chi) = (\psi, \chi) + \frac{1}{3}(\psi', \chi'), \quad \forall \psi, \chi \in S_{h,0}^4,$$

and $R_h : H^1 \rightarrow S_{h,0}^4$ be the associated elliptic projection operator defined by

$$a(R_h v, \chi) = a(v, \chi), \quad \forall \chi \in S_{h,0}^4.$$

It follows by standard estimates that if $1 \leq k \leq 4$

$$\|R_h w - w\|_j \leq Ch^{k-j} \|w\|_k \quad j = 0, 1, \quad \text{if } w \in H^k \cap H_0^1, \quad (3.2)$$

and that, [19],

$$\|R_h w - w\|_\infty + h \|R_h w - w\|_{1,\infty} \leq Ch^4 \|w\|_{4,\infty} \quad \text{if } w \in W^{4,\infty} \cap H_0^1. \quad (3.3)$$

Similar estimates hold for the analogous elliptic projection into S_h^4 . In addition, as a consequence of the quasiuniformity of the mesh, the inverse inequalities

$$\begin{aligned} \|\chi\|_\beta &\leq Ch^{-(\beta-\alpha)} \|\chi\|_\alpha, \quad 0 \leq \alpha \leq \beta \leq 2, \\ \|\chi\|_{s,\infty} &\leq Ch^{-(s+1/2)} \|\chi\|, \quad 0 \leq s \leq 2, \end{aligned} \quad (3.4)$$

hold for any $\chi \in S_h^4$ (or any $\chi \in S_{h,0}^4$), and so does the estimate, [19],

$$\|Pv - v\|_\infty \leq Ch^4 \|v\|_{4,\infty}, \quad v \in W^{4,\infty}, \quad (3.5)$$

where $P : L^2 \rightarrow S_h^4$ is the L^2 -projection operator onto S_h^4 . As a consequence of the approximation and inverse properties of the cubic spline spaces we also have that P is stable in L^∞ and in H^1 , and that R_h is stable in H_0^1 and in $H^2 \cap H_0^1$.

We let the *standard Galerkin semidiscretization* on S_h^4 of (CB) be defined as follows: We seek $\eta_h : [0, T] \rightarrow S_h^4$, $u_h : [0, T] \rightarrow S_{h,0}^4$, such that for $t \in [0, T]$

$$\begin{aligned} (\eta_{ht}, \phi) + (u_{hx}, \phi) + ((\eta_h u_h)_x, \phi) &= 0 \quad \forall \phi \in S_h^4, \\ a(u_{ht}, \chi) + (\eta_{hx}, \chi) + (u_h u_{hx}, \chi) &= 0 \quad \forall \chi \in S_{h,0}^4, \end{aligned} \quad (3.6)$$

with initial conditions

$$\eta_h(0) = P\eta_0, \quad u_h(0) = R_h u_0. \quad (3.7)$$

We also similarly define the analogous semidiscretization of (SCB) which is given for $t \in [0, T]$ by

$$\begin{aligned} (\eta_{ht}, \phi) + (u_{hx}, \phi) + \frac{1}{2}((\eta_h u_h)_x, \phi) &= 0 \quad \forall \phi \in S_h^4, \\ a(u_{ht}, \chi) + (\eta_{hx}, \chi) + \frac{3}{2}(u_h u_{hx}, \chi) + \frac{1}{2}(\eta_h \eta_{hx}, \phi) &= 0 \quad \forall \chi \in S_{h,0}^4, \end{aligned} \quad (3.8)$$

with

$$\eta_h(0) = P\eta_0, \quad u_h(0) = R_h u_0. \quad (3.9)$$

The following result may be proved in a totally analogous manner to the analogous result of Proposition 2.1 and, hence, its proof will be omitted.

Proposition 3.1. *Let h be sufficiently small. Suppose that the solutions of (CB) and (SCB) are such that $\eta \in C(0, T; W_\infty^4)$, $u \in C(0, T; W_\infty^4 \cap H_0^1)$. Then, the semidiscrete problems (3.6), (3.7) and (3.8), (3.9) have unique solutions (η_h, u_h) for $0 \leq t \leq T$ that satisfy*

$$\max_{0 \leq t \leq T} \|\eta(t) - \eta_h(t)\| \leq Ch^3, \quad (3.10)$$

$$\max_{0 \leq t \leq T} \|u(t) - u_h(t)\|_1 \leq Ch^3. \quad (3.11)$$

3.2. Uniform mesh. For integer $N \geq 2$ we let $h = 1/N$ and $x_i = (i-1)h$, $i = 1, 2, \dots, N+1$. In this section we will denote by $\{\phi_j\}_{j=1}^{N+3}$ the usual B -spline basis of S_h^4 defined by the restrictions on $[0, 1]$ of the functions $\phi_j(x) = \Phi(\frac{x}{h} - (j-2))$, where Φ is the cubic spline on \mathbb{R} with respect to the partition $\{-2, -1, 0, 1, 2\}$ with support $[-2, 2]$ and nodal values $\Phi(0) = 1$, $\Phi(\pm 1) = 1/4$, $\Phi(\pm 2) = 0$. Thus, e.g. $\text{supp}(\phi_j) = [x_{j-3}, x_{j+1}]$ and $\phi_j(x_{j-1}) = 1$ for $4 \leq j \leq N$, etc. Our aim in this paragraph is to prove that the cubic spline standard Galerkin semidiscretizations of the two systems on this mesh satisfy the estimates $\|\eta - \eta_h\| = O(h^{3.5} \sqrt{\ln 1/h})$, $\|u - u_h\|_1 = O(h^3)$, $\|u - u_h\| = O(h^4 \sqrt{\ln 1/h})$, $\|u_t - u_{ht}\| = O(h^4 \sqrt{\ln 1/h})$. For this purpose we shall first state and prove a series of auxiliary results.

Our first lemma is a well known result, the cubic spline analog of Lemma 2.1.

Lemma 3.1. (i) *Let $G_{ij} = (\phi_j, \phi_i)$, $1 \leq i, j \leq N+3$. Then, there exist positive constants c_1 and c_2 such that*

$$c_1 h |\gamma|^2 \leq \langle G\gamma, \gamma \rangle \leq c_2 h |\gamma|^2 \quad \forall \gamma \in \mathbb{R}^{N+3}.$$

(ii) *Let $b \in \mathbb{R}^{N+3}$, $\gamma = G^{-1}b$, and $\psi = \sum_{j=1}^{N+3} \gamma_j \psi_j$. Then*

$$\|\psi\| \leq (c_1 h)^{-1/2} |b|.$$

(iii) *Let $w \in C^5$. Then, there exists a constant $C_1 = C_1(\|w^{(5)}\|_\infty)$ such that for any $\hat{x} \in [x_i, x_{i+1}]$,*

$$(w - I_h w)(x) = \frac{1}{4!} w^{(4)}(\hat{x})(x - x_i)^2(x_{i+1} - x)^2 + \tilde{g}(x), \quad x_i \leq x \leq x_{i+1},$$

where $\|\tilde{g}\|_\infty + h\|\tilde{g}'\|_\infty \leq C_1 h^5$.

Proof. The proofs of (i)-(iii) are given in [20] in the case of periodic cubic splines. It is straightforward to adapt them to the case of S_h^4 and I_h at hand. \square

We next prove a superaccuracy estimate for $P[(w\varepsilon)']$, where ε is the error of the cubic spline interpolant of a sufficiently smooth function and w is a C^1 weight. This estimate is a consequence of cancellation effects due to the uniform mesh and may be viewed as the cubic spline analog of Lemma 2.2.

Lemma 3.2. *Let $v \in C^5$ and $w \in C^1$. If $\varepsilon = v - I_h v$ and $\psi \in S_h^4$ is such that*

$$(\psi, \phi) = ((w\varepsilon)', \phi) \quad \forall \phi \in S_h^4,$$

then $\|\psi\| \leq Ch^{3.5}$. If in addition $w(0) = w(1) = 0$, then $\|\psi\| \leq Ch^4$.

Proof. Let $b_j = ((w\varepsilon)', \phi_j) = -(w\varepsilon, \phi_j')$, $1 \leq j \leq N+3$. In view of Lemma 3.1(ii) it suffices to show that $|b| = O(h^4)$. It is clear by (3.1) and the properties of the basis functions ϕ_j that $b_j = O(h^4)$ for all j . We

will prove that actually $b_j = O(h^5)$ for $j = 4, 5, \dots, N$, thus establishing that $|b| = O(h^4)$. Let $4 \leq j \leq N$. Using Lemma 3.1(iii) and putting $q(x) = x^2(h-x)^2/4!$ we have

$$\begin{aligned}
(w\varepsilon, \phi'_j) &= \sum_{k=0}^3 \int_{x_{j-3+k}}^{x_{j-2+k}} w(x)\varepsilon(x)\phi'_j(x)dx \\
&= v^{(4)}(x_{j-2}) \int_{x_{j-3}}^{x_{j-2}} w(x)q(x-x_{j-3})\phi'_j(x)dx + v^{(4)}(x_{j-1}) \int_{x_{j-2}}^{x_{j-1}} w(x)q(x-x_{j-2})\phi'_j(x)dx \\
&\quad + v^{(4)}(x_{j-1}) \int_{x_{j-1}}^{x_j} w(x)q(x-x_{j-1})\phi'_j(x)dx + v^{(4)}(x_j) \int_{x_j}^{x_{j+1}} w(x)q(x-x_j)\phi'_j(x)dx + O(h^5) \\
&=: v^{(4)}(x_{j-2})J_1 + v^{(4)}(x_{j-1})(J_2 + J_3) + v^{(4)}(x_j)J_4 + O(h^5).
\end{aligned}$$

Since $v^{(4)}(x_{j-2}) = v^{(4)}(x_{j-1}) - hv^{(5)}(t_{1j})$ and $v^{(4)}(x_j) = v^{(4)}(x_{j-1}) + hv^{(5)}(t_{2j})$ for some $t_{1j} \in (x_{j-2}, x_{j-1})$ and $t_{2j} \in (x_{j-1}, x_j)$, we obtain

$$(w\varepsilon, \phi'_j) = v^{(4)}(x_{j-1})(J_1 + J_2 + J_3 + J_4) + O(h^5). \quad (3.12)$$

Suitable changes of variable in each of the four integrals yield

$$\begin{aligned}
J_1 &= \int_0^h w(x+x_{j-3})q(x)\phi'_4(x)dx, & J_2 &= \int_0^h w(x+x_{j-2})q(x)\phi'_3(x)dx, \\
J_3 &= \int_0^h w(x+x_{j-1})q(x)\phi'_2(x)dx, & J_4 &= \int_0^h w(x+x_j)q(x)\phi'_1(x)dx.
\end{aligned}$$

In addition, if $x_{j\mu} = (x_{j-2} + x_{j-1})/2$, we have for $x \in [0, h]$

$$\begin{aligned}
w(x+x_{j-3}) &= w(x+x_{j\mu}) - \frac{3h}{2}w'(\tau_{1jx}), & w(x+x_{j-2}) &= w(x+x_{j\mu}) - \frac{h}{2}w'(\tau_{2jx}), \\
w(x+x_{j-1}) &= w(x+x_{j\mu}) + \frac{h}{2}w'(\tau_{3jx}), & w(x+x_j) &= w(x+x_{j\mu}) + \frac{3h}{2}w'(\tau_{4jx}),
\end{aligned}$$

for some τ_{ijx} between $x+x_{j-4+i}$ and $x+x_{j\mu}$, $i = 1, 2, 3, 4$. Thus,

$$\begin{aligned}
J_1 + J_2 + J_3 + J_4 &= \int_0^h w(x+x_{j\mu})q(x)[\phi'_1(x) + \phi'_2(x) + \phi'_3(x) + \phi'_4(x)]dx + O(h^5) \\
&= w(x_{j\mu}) \int_0^h q(x)[\phi'_1(x) + \phi'_2(x) + \phi'_3(x) + \phi'_4(x)]dx + O(h^5).
\end{aligned}$$

The last integral is equal to zero, since $\phi_4(x) = \phi_1(h-x)$, $\phi_3(x) = \phi_4(h-x)$, and $q(x) = q(h-x)$ for $x \in [0, h]$. Hence

$$J_1 + J_2 + J_3 + J_4 = O(h^5),$$

and (3.12) implies that $b_j = O(h^5)$. Thus, the first assertion of the lemma is verified. To prove the second assertion, suppose that $w(0) = 0$. Then $b_j = O(h^5)$ for $j = 1, 2, 3$. Indeed, using Lemma 3.1(iii), we have

$$\begin{aligned}
-b_1 &= (w\varepsilon, \phi'_1) = v^{(4)}(h) \int_0^h w(x)q(x)\phi'_1(x)dx + O(h^5) \\
&= v^{(4)}(h) \int_0^h xw'(t_x)q(x)\phi'_1(x)dx + O(h^5) = O(h^5).
\end{aligned}$$

In addition,

$$\begin{aligned}
-b_2 &= (w\varepsilon, \phi'_2) = \int_0^h w(x)\varepsilon(x)\phi'_2(x)dx + \int_h^{2h} w(x)\varepsilon(x)\phi'_2(x)dx \\
&= v^{(4)}(h) \int_0^h w(x)q(x)\phi'_2(x)dx + v^{(4)}(h) \int_h^{2h} w(x)q(h-x)\phi'_2(x)dx + O(h^5) \\
&= v^{(4)}(h) \int_0^h w(x)q(x)\phi'_2(x)dx + v^{(4)}(h) \int_0^h w(x+h)q(x)\phi'_1(x)dx + O(h^5) \\
&= v^{(4)}(h) \int_0^h xw'(t_x)q(x)\phi'_2(x)dx + v^{(4)}(h) \int_0^h (w(x+h) - w(x))q(x)\phi'_1(x)dx \\
&= -b_1 + O(h^5) = O(h^5).
\end{aligned}$$

Finally,

$$\begin{aligned}
-b_3 &= (w\varepsilon, \phi'_3) = \int_0^h w(x)\varepsilon(x)\phi'_3(x)dx + \int_h^{2h} w(x)\varepsilon(x)\phi'_3(x)dx + \int_{2h}^{3h} w(x)\varepsilon(x)\phi'_3(x)dx \\
&= v^{(4)}(h) \left[\int_0^h w(x)q(x)\phi'_3(x)dx + \int_0^h w(x+h)q(x)\phi'_2(x)dx \right. \\
&\quad \left. + \int_0^h w(x+2h)q(x)\phi'_1(x)dx \right] + O(h^5) \\
&= v^{(4)}(h) \left[\int_0^h xw'(t_x)q(x)\phi'_3(x)dx - b_1 - b_2 \right] + O(h^5) = O(h^5).
\end{aligned}$$

Similarly, if $w(1) = 0$ we have $b_j = O(h^5)$ for $j = N+1, N+2, N+3$. Hence, if w vanishes at $x = 0$ and $x = 1$, $|b| = O(h^5)$ and the second assertion of the lemma follows by Lemma 3.1(ii). \square

We shall also derive a superaccuracy estimate for $P[(we)']$, where e is the error of the elliptic projection of a function $v \in C_0^5$ and w is a C^1 weight. For this purpose, we first state two superconvergence results that follow from the analysis of Wahlbin in [30], and which are valid for interior nodes whose distance from the endpoints of the interval is at least of $O(h \ln 1/h)$.

Proposition 3.2. *Suppose that $v \in C_0^5$ and let $v_h = R_h v$ be its elliptic projection onto $S_{h,0}^4$. Then the following hold:*

(i) *There exists a constant C independent of h such that*

$$|(v - v_h)'(x_i)| \leq Ch^4 \|v\|_{W_\infty^5} \quad \text{provided} \quad \text{dist}(x_i, \partial I) \geq C_1 h \ln \frac{1}{h}, \quad (3.13)$$

where C_1 is a sufficiently large constant independent of h .

(ii) *If x_i, x_{i+1} are two adjacent nodes for which the second inequality in (3.13) holds and $e(x) = v(x) - v_h(x)$, we have*

$$e(x_{i+1}) - e(x_i) = O(h^5) \|v\|_{W_\infty^5}. \quad (3.14)$$

Proof. (i) The estimate (3.13) follows from Corollary 1.6.2 of [30] (which is strictly valid when the elliptic projection is defined by $(\tilde{v}'_h, \phi') = (v', \phi') \forall \phi \in S_{h,0}^4$), combined with Theorem 1.3.1 of [30] that allows us to state the result for $v_h = R_h v$ defined as in section 2 of the paper at hand.

(ii) The cancellation property expressed by (3.14) is a consequence of the fact that $\tilde{e}(x) = v(x) - \tilde{v}_h(x)$ may be represented in the form

$$\tilde{e}(x) = Q_i(x) + O(h^5) \|w\|_{W_\infty^5(x_i, x_{i+1})}, \quad x \in [x_i, x_{i+1}],$$

where $Q_i(x)$ is a polynomial of degree four such that $Q_i(x_i) = Q_i(x_{i+1})$. This representation follows from the remarks in Example 1.8.2 of [30] and by adapting the arguments of the proof in Section 1.8 of [30] (which are valid for C^1 Hermite cubics) to the case of the C^2 cubic splines at hand. Hence $\tilde{e}(x_{i+1}) - \tilde{e}(x_i) = O(h^5) \|w\|_{W_\infty^5(x_i, x_{i+1})}$, and (3.14) follows by the function values superconvergence estimate for elliptic projections given in Theorem 1.3.2 of [30]. \square

In the next lemma we present some further formulas for the error $v - R_h v$ that will be used in the sequel.

Lemma 3.3. (i) Let $v \in C_0^5$, $v_h = R_h v$, $e = v - v_h$, and $1 \leq i \leq N$. Then, there exists a constant $C = C(\|w\|_{W_\infty^5})$ such that for any $\hat{x} \in [x_i, x_{i+1}]$,

$$e(x) = \gamma_i(x) + \frac{1}{4!} v^{(4)}(\hat{x})(x - x_i)^2(x_{i+1} - x)^2 + \delta_i(x), \quad x \in [x_i, x_{i+1}], \quad (3.15)$$

where $\|\delta_i\|_\infty \leq Ch^5$ and γ_i is the cubic Hermite polynomial interpolating the values of e and its derivative at the nodes x_i and x_{i+1} .

(ii) In addition to the hypotheses in (i), suppose that x_i and x_{i+1} satisfy the second inequality in (3.13). Then, there is a constant $C = C(\|v\|_{W_\infty^5})$ such that for any $\hat{x} \in [x_i, x_{i+1}]$

$$e(x) = e(x_i) + \frac{1}{4!} v^{(4)}(\hat{x})(x - x_i)^2(x_{i+1} - x)^2 + \tilde{\delta}_i(x), \quad x \in [x_i, x_{i+1}], \quad (3.16)$$

where $\|\tilde{\delta}_i\|_\infty \leq Ch^5$.

Proof. (i) By the standard representation of the error of Hermite interpolation we have, since $v_h \in \mathbb{P}_3$ in $[x_i, x_{i+1}]$, that for $x \in [x_i, x_{i+1}]$ there holds

$$e(x) - \gamma_i = \frac{1}{4!} (x - x_i)^2(x_{i+1} - x)^2 v^{(4)}(t_x)$$

for some $t_x \in (x_i, x_{i+1})$. Hence (3.15) follows from the mean-value theorem since $v \in C^5$.

(ii) We let $I = [0, 1]$, and $K = \{x \in I : \text{dist}(x, \partial I) \geq C_1 h \ln 1/h\}$. In view of (3.15) it suffices to show that

$$\gamma_i(x) = e(x_i) + O(h^5) \|v\|_{W_\infty^5} \quad \text{for } x_i, x_{i+1} \in K.$$

Now

$$\gamma_i(x) = e(x_i)A_{i,1}(x) + e'(x_i)B_{i,1}(x) + e(x_{i+1})A_{i,2}(x) + e'(x_{i+1})B_{i,2}(x),$$

where

$$\begin{aligned} A_{i,1}(x) &= \left[1 + \frac{2(x-x_i)}{h}\right] \frac{(x_{i+1}-x)^2}{h^2}, & B_{i,1}(x) &= \frac{(x-x_i)(x_{i+1}-x)^2}{h^2}, \\ A_{i,2}(x) &= \left[1 + \frac{2(x_{i+1}-x)}{h}\right] \frac{(x-x_i)^2}{h^2}, & B_{i,2}(x) &= \frac{-(x_{i+1}-x)(x-x_i)^2}{h^2}. \end{aligned}$$

Hence, using (3.13) and (3.14) we see that

$$\begin{aligned} \gamma_i(x) &= e(x_i)A_{i,1}(x) + e(x_{i+1})A_{i,2}(x) + O(h^5) \|v\|_{W_\infty^5} \\ &= e(x_i)(A_{i,1}(x) + A_{i,2}(x)) + O(h^5) \|v\|_{W_\infty^5} \\ &= e(x_i) + O(h^5) \|v\|_{W_\infty^5}, \end{aligned}$$

which concludes the proof of the lemma. \square

We are now ready to prove the required estimate for $P[(we)']$, where w is a C^1 weight.

Lemma 3.4. Let $v \in C_0^5$ and $w \in C^1$. If $e = v - R_h v$ and $\psi \in S_h^4$ is such that

$$(\psi, \phi) = ((we)', \phi) \quad \forall \phi \in S_h^4,$$

then $\|\psi\| \leq Ch^{3.5} \sqrt{\ln 1/h}$.

Proof. Let $b_j = ((we)', \phi_j)$, $1 \leq j \leq N+3$. In view of Lemma 3.1(ii) it suffices to show that $|b_j| = O(h^4 \sqrt{\ln 1/h})$. It is clear by (3.2) and the properties of the basis functions ϕ_j that $b_j = O(h^4)$. We will show that for ‘most’ of the indices j it is true that $b_j = O(h^5)$. Let $I = [0, 1]$ and $K = \{x \in I : \text{dist}(x, \partial I) \geq C_1 h \ln 1/h\}$. Since $x_i = (i-1)h$, $1 \leq i \leq N+1$, it is clear that if $x_i \in K$ then $1 + C_1 \ln 1/h \leq i \leq N+1 - C_1 \ln 1/h$. Therefore $x_i \in K$ if and only if $i \in M = \{i \in \mathbb{N} : i \in [1 + C_1 \ln 1/h, N+1 - C_1 \ln 1/h]\}$. Let $\mu = \min M$ and $m = \max M$. We shall show that

$$|b_j| \leq Ch^5, \quad j = \mu + 4, \dots, m-1. \quad (3.17)$$

Indeed, for such j we have

$$\begin{aligned} -b_j &= (we, \phi_j') = \sum_{k=0}^3 \int_{x_{j-3+k}}^{x_{j-2+k}} w(x) e(x) \phi_j'(x) dx \\ &= \sum_{k=0}^3 \left\{ e(x_{j-3+k}) \int_{x_{j-3+k}}^{x_{j-2+k}} w(x) \phi_j'(x) dx + \int_{x_{j-3+k}}^{x_{j-2+k}} (e(x) - e(x_{j-3+k})) w(x) \phi_j'(x) dx \right\}. \end{aligned}$$

Using now Lemma 3.3(ii) in each one of the second group of integrals of this expression and similar considerations as in the proof of Lemma 3.2, we see that

$$-b_j = e(x_{j-3})J_1 + e(x_{j-2})J_2 + e(x_{j-1})J_3 + e(x_j)J_4 + O(h^5),$$

where $J_i = \int_{x_{j-4+i}}^{x_{j-3+i}} w(x)\phi'_j(x)dx$, $1 \leq i \leq 4$. But since $e(x_{i+1}) = e(x_i) + O(h^5)$ when $x_i, x_{i+1} \in K$ (by Proposition 3.2(ii)), and taking into account that the J_i are bounded independently of h and that $J_1 + J_2 + J_3 + J_4 = O(h)$ (by similar considerations as in the proof of Lemma 3.2), we finally obtain (3.17). By the definitions of μ and m we now have

$$\begin{aligned} |b|^2 &= \sum_{j=1}^{\mu+3} b_j^2 + \sum_{j=\mu+4}^{m-1} b_j^2 + \sum_{j=m}^{N+3} b_j^2 \leq C[(\mu+3)h^8 + h^9 + (N+4-m)h^8] \\ &\leq C[(C_2 + C_1 \ln \frac{1}{h})h^8 + h^9] \leq C_3 h^8 \ln \frac{1}{h}, \end{aligned}$$

where $C_3 = C_3(\|v\|_{W_\infty^5}, \|w\|_{W_\infty^1}, C_1)$, thus concluding the proof of the lemma. \square

The main result of this section follows.

Theorem 3.1. *Let $h = 1/N$ be sufficiently small. Suppose that the solutions of (CB) and (SCB) are such that $\eta \in C(0, T; C^5)$, $\eta_t \in C(0, T; C^4)$, $u, u_t \in C(0, T; C_0^5)$. Then the semidiscrete problems (3.6) and (3.8) with initial conditions*

$$\eta_h(0) = I_h \eta_0, \quad u_h(0) = R_h u_0, \quad (3.18)$$

have unique solutions (η_h, u_h) for $0 \leq t \leq T$ that satisfy

$$\begin{aligned} (i) \quad & \max_{0 \leq t \leq T} \|\eta(t) - \eta_h(t)\| \leq Ch^{3.5} \sqrt{\ln \frac{1}{h}}, \quad \max_{0 \leq t \leq T} \|u(t) - u_h(t)\|_1 \leq Ch^3, \\ (ii) \quad & \max_{0 \leq t \leq T} \|u(t) - u_h(t)\| \leq Ch^4 \sqrt{\ln \frac{1}{h}}, \quad \max_{0 \leq t \leq T} \|u_t(t) - u_{ht}(t)\| \leq Ch^4 \sqrt{\ln \frac{1}{h}}. \end{aligned}$$

Proof. Again, we give the proof in the case of (SCB), noting that the analogous proof for (CB) follows as in Proposition 2.1. The technique of proof is basically the same as the one used in Theorem 2.1 and we shall give here the details that are different. We define now

$$\rho := \eta - I_h \eta, \quad \theta := I_h \eta - \eta_h, \quad \sigma := u - R_h u, \quad \xi := R_h u - u_h,$$

and, arguing as in the proof of Theorem 2.1, we have for $0 \leq t \leq T$

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|\theta\|^2 + \|\xi\|_1^2) + (((1 + \frac{1}{2}\eta)\sigma)_x, \theta) + \frac{1}{2} ((u\theta)_x, \theta) + \frac{1}{2} (F_x, \theta) \\ & + (\rho_x, \xi) + \frac{1}{2} ((\eta\theta)_x, \xi) + \frac{1}{2} (G_x, \xi) + \frac{3}{2} (H_x, \xi) \\ & = -(\rho_t, \theta), \end{aligned} \quad (3.19)$$

where

$$\begin{aligned} F &:= \eta\xi + u\rho - \rho\sigma - \rho\xi - \theta\sigma, \\ G &:= \eta\rho - \rho\theta - \frac{1}{2}\rho^2, \\ H &:= u\sigma + u\xi - \sigma\xi - \frac{1}{2}\sigma^2 - \frac{1}{2}\xi^2. \end{aligned}$$

Using the approximation properties of S_h^4 and $S_{h,0}^4$, integration by parts, and Lemmas 3.2 and 3.4 we have

$$\begin{aligned} |(((1 + \frac{1}{2}\eta)\sigma)_x, \theta)| &\leq Ch^{3.5} \sqrt{\ln \frac{1}{h}} \|\theta\|, & |((u\theta)_x, \theta)| &\leq C\|\theta\|^2, \\ |(F_x, \theta)| &\leq C(\|\xi\|_1 \|\theta\| + h^4 \|\theta\| + \|\theta\|^2), & |(\rho_x, \xi)| &\leq Ch^4 \|\xi\|_1, \\ |((\eta\theta)_x, \xi)| &\leq C\|\xi\|_1 \|\theta\|, & |(G_x, \xi)| &\leq C(\|\xi\|_1 \|\theta\| + h^4 \|\xi\|_1), \\ |(H_x, \xi)| &\leq C(\|\xi\|^2 + h^4 \|\xi\|_1), & |(\rho_t, \theta)| &\leq Ch^4 \|\theta\|. \end{aligned}$$

(Here and in the rest of the proof of this theorem we have omitted details that are similar to those in the analogous steps of the proof of Theorem 2.1.) Therefore, (3.19) gives for $0 \leq t \leq T$

$$\frac{d}{dt} (\|\theta\|^2 + \|\xi\|_1^2) \leq C(h^7 \ln \frac{1}{h} + \|\theta\|^2 + \|\xi\|_1^2),$$

from which, by Gronwall's Lemma and (3.18) we get

$$\|\theta\| + \|\xi\|_1 \leq Ch^{3.5} \sqrt{\ln \frac{1}{h}}, \quad (3.20)$$

and the estimates (i) follow.

In order to prove the estimates (ii), we first note that that

$$\|F\| \leq C(\|\xi\| + h^4), \quad \|G\| \leq Ch^4, \quad \|H\| \leq C(\|\xi\| + h^4).$$

Noting that $\xi_t = R_h v$, where v is the solution of the problem

$$\begin{aligned} v - \frac{1}{3}v'' &= -(\theta + \rho)_x - \frac{1}{2}(\eta\theta)_x - \frac{1}{2}(G - \frac{1}{2}\theta^2 + 3H)_x, \quad x \in [0, 1], \\ v(0) &= v(1) = 0, \end{aligned}$$

we see by (3.20) that $\|v\|_1 \leq Ch^{3.5} \sqrt{\ln 1/h}$. Arguing as in the proof of Theorem 2.1 we have now $\|v\| \leq C(h^4 + \|\gamma\| + \|\xi\|)$, where $\gamma = \int_0^x (\theta(s, t) + \rho(s, t)) ds - x \int_0^1 \rho(s, 0) ds$. Then there follows that

$$\frac{1}{2} \frac{d}{dt} \|\xi\|^2 \leq C(h^8 + \|P_0 \gamma\|^2 + \|\xi\|^2),$$

where P_0 is the L^2 -projection operator onto S_h^3 , the space of C^1 piecewise quadratic functions relative to the partition $\{x_j\}$. We also have, in view of (3.20) and the estimate $\|P_0 \gamma - \gamma\| \leq Ch\|\gamma\|_1$, that

$$\frac{1}{2} \frac{d}{dt} \|P_0 \gamma\|^2 \leq C(h^8 \ln \frac{1}{h} + \|P_0 \gamma\|^2 + \|\xi\|^2).$$

From the last two inequalities and Gronwall's Lemma we obtain

$$\|P_0 \gamma\| + \|\xi\| \leq Ch^4 \sqrt{\ln \frac{1}{h}},$$

from which the first estimate of (ii) follows. The second estimate also follows along the lines of the proof of Theorem 2.1. \square

Remark 3.1. The results of the Theorem also hold if we take as $\eta_h(0)$ any other approximation of η_0 in S_h^4 of optimal order of accuracy in L^2 .

Remark 3.2. The superaccuracy estimate $\|\xi\|_1 = O(h^{3.5} \sqrt{\ln 1/h})$ of (3.20), combined with (3.3) and Sobolev's inequality yield the L^∞ estimate $\|u - u_h\|_\infty = O(h^{3.5} \sqrt{\ln 1/h})$.

3.3. Numerical experiments. We considered the nonhomogeneous (SCB) system and we discretized it on a uniform mesh with diminishing $h = 1/N$ on $[0, 1]$ using cubic splines for the spatial discretization and fourth-order accurate explicit Runge Kutta time stepping, (cf. Section 4), with time step $k = h/10$ for which the temporal discretization error was negligible in comparison with the spatial error. We took a suitable right-hand side so that the exact solution of the system was $\eta = \exp(2t)(\cos(\pi x) + x + 2)$, $u = \exp(xt)(\sin(\pi x) + x^3 - x^2)$. The errors and orders of convergence produced by this numerical experiments are shown in Table 3.1. The rates are close to the theoretical predictions of Theorem 3.1. The table suggests that the L^2 rate of convergence for η is slightly less than 3.5, while that for u is essentially four. It further suggests that $\|\eta - \eta_h\|_1 = O(h^{2.5})$, $\|u - u_h\|_1 = O(h^3)$ (agreeing with the second estimate of (i) of Theorem 3.1), $\|\eta - \eta_h\|_\infty = O(h^3)$ and $\|u - u_h\|_\infty = O(h^4)$. We also mention that the $W^{1,\infty}$ orders of convergence (not shown here) were approximately equal to 2.2 for η and 3 for u , and that the convergence rates from a similar experiment with (CB) were practically the same. In Figure 3.1 we plot the quantity $\kappa := \|\eta - \eta_h\|/(h^{3.5} \sqrt{\ln 1/h})$ as a function of $N = 1/h$; here $\|\eta - \eta_h\|$ are the L^2 -errors from Table 3.1. We observe that κ apparently approaches a constant close to 0.13 as N grows, which seems to be consistent with the presence of a slow-varying modulation of $h^{3.5}$ as $h \rightarrow 0$, such as $(\ln 1/h)^{1/2}$. We close this paragraph with a remark on the 'effect of the boundary' on the error estimates of Theorem 3.1. The proofs of Lemma 3.4 and Theorem 3.1 suggest that the accuracy of ψ in Lemma 3.4 and e.g. of $\|\eta - \eta_h\|$ in Theorem 3.1 degenerates near the boundary of the interval. This is consistent with the results of the following numerical experiment. We integrated in time the (SCB) system on $[0, 1]$ with suitable right-hand side and initial conditions so that the wave elevation is given by the travelling Gaussian profile $\eta(x, t) = 0.5 \exp[-144(x - 0.5 - 0.2t)^2]$ and the velocity by $u(x, t) = 6(\sqrt{\eta + 1} - 1)x(x - 1)$. (We use cubic splines in space and the explicit fourth-order RK scheme in time.) The support of the initial η -profile is effectively contained in the interval $[0.3, 0.5]$ and the wave moves to the right and starts crossing the boundary at $x = 1$ at about $t = 1.5$ (see Figure 3.2). In

L^2 -errors

N	η	order	u	order
40	0.8063(-6)		0.8032(-7)	
80	0.7178(-7)	3.490	0.5062(-8)	3.988
120	0.1744(-7)	3.489	0.1003(-8)	3.993
160	0.6393(-8)	3.489	0.3178(-9)	3.994
200	0.2934(-8)	3.490	0.1303(-9)	3.996
240	0.1553(-8)	3.490	0.6288(-10)	3.996
280	0.9068(-9)	3.490	0.3395(-10)	3.998
320	0.5691(-9)	3.490	0.1986(-10)	4.015
360	0.3773(-9)	3.490	0.1238(-10)	4.012
400	0.2612(-9)	3.489	0.8106(-11)	4.021
440	0.1873(-9)	3.489	0.5533(-11)	4.008
480	0.1382(-9)	3.492	0.4011(-11)	3.696
520	0.1046(-9)	3.488	0.2840(-11)	4.315

 L^∞ -errors

N	η	order	u	order
40	0.2666(-5)		0.2456(-6)	
80	0.3611(-6)	2.884	0.1590(-7)	3.949
120	0.1100(-6)	2.931	0.3175(-8)	3.973
160	0.4708(-7)	2.951	0.1010(-8)	3.982
200	0.2431(-7)	2.962	0.4150(-9)	3.986
240	0.1415(-7)	2.969	0.2005(-9)	3.989
280	0.8947(-8)	2.973	0.1084(-9)	3.990
320	0.6010(-8)	2.980	0.6367(-10)	3.986
360	0.4230(-8)	2.982	0.3980(-10)	3.988
400	0.3088(-8)	2.986	0.2616(-10)	3.982
440	0.2323(-8)	2.988	0.1790(-10)	3.985
480	0.1797(-8)	2.950	0.1256(-10)	4.070
520	0.1412(-8)	3.014	0.9168(-11)	3.932

 H^1 -errors

N	η	order	u	order
40	0.1298(-3)		0.2014(-4)	
80	0.2215(-4)	2.550	0.2540(-5)	2.987
120	0.7923(-5)	2.536	0.7550(-6)	2.992
160	0.3829(-5)	2.528	0.3190(-6)	2.995
200	0.2182(-5)	2.521	0.1635(-6)	2.996
240	0.1379(-5)	2.516	0.9467(-7)	2.997
280	0.9363(-6)	2.512	0.5964(-7)	2.997
320	0.6699(-6)	2.508	0.3997(-7)	2.998
360	0.4987(-6)	2.506	0.2808(-7)	2.998
400	0.3831(-6)	2.503	0.2047(-7)	2.998
440	0.3018(-6)	2.502	0.1538(-7)	2.998
480	0.2428(-6)	2.500	0.1185(-7)	2.998
520	0.1988(-6)	2.499	0.9323(-8)	2.999

TABLE 3.1. Errors and orders of convergence. (SCB) system, standard Galerkin semidiscretization with cubic splines on a uniform mesh.

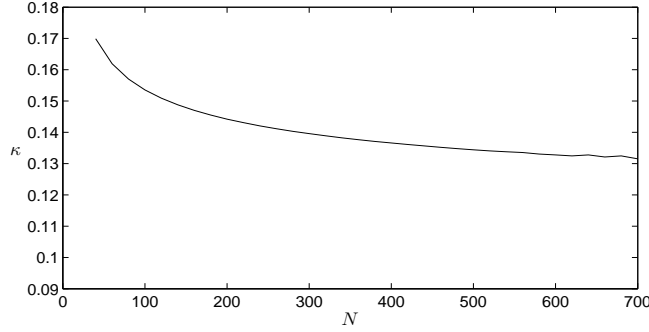
FIGURE 3.1. $\kappa := \|\eta - \eta_h\|/h^{3.5}\sqrt{\ln 1/h}$ as a function of $N = 1/h$; the $\|\eta - \eta_h\|$ are the L^2 errors from Table 3.1.

Table 3.2 we show the L^2 errors of η , as $N = 1/h$ increases, at the temporal instances $t = 0.5, 1.0, 1.5, 2.0$ and 2.5 . The rates of convergence are practically equal to four up to $t = 1.5$ but as η becomes nonzero at the boundary they fall to a value consistent with the first inequality of (i) of Theorem 3.1.

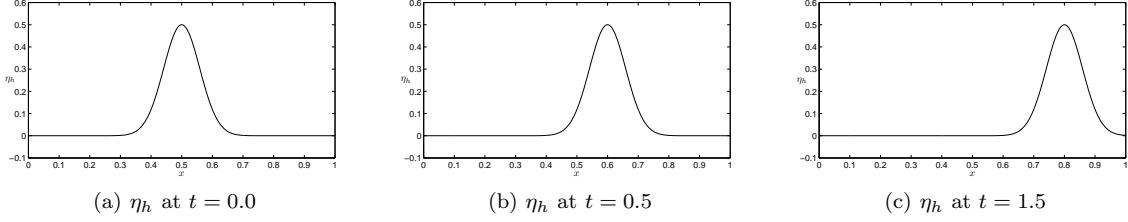


FIGURE 3.2. Travelling Gaussian η -profile. Nonhomogeneous (SCB) system.

time	0.5		1.0		1.5		2.0		2.5	
N	L^2 -error	order	L^2 -error	order	L^2 -error	order	L^2 -error	order	L^2 -error	order
250	1.3314(-08)		1.0661(-08)		1.3596(-08)		1.5924(-08)		1.9906(-08)	
500	8.2780(-10)	4.008	6.6223(-10)	4.009	8.4585(-10)	4.007	1.0596(-09)	3.910	1.7594(-09)	3.500
750	1.6334(-10)	4.003	1.3067(-10)	4.003	1.6706(-10)	4.000	2.2223(-10)	3.852	4.2637(-10)	3.496
1000	5.1617(-11)	4.004	4.1350(-11)	4.000	5.2838(-11)	4.001	7.4176(-11)	3.814	1.5595(-10)	3.496
1250	2.1179(-11)	3.992	1.6922(-11)	4.004	2.1710(-11)	3.986	3.1966(-11)	3.772	7.1471(-11)	3.497
1500	1.0287(-11)	3.961	8.1703(-12)	3.994	1.0554(-11)	3.956	1.6213(-11)	3.724	3.7803(-11)	3.493

TABLE 3.2. L^2 -errors of η and orders of convergence. Example of Figure 3.2.

4. FULLY DISCRETE SCHEMES

In this section we turn to the study of some temporal discretizations of the o.d.e. systems represented by the standard Galerkin spatial discretizations of (CB) or (SCB), such as (2.2) or (2.4), for example. We shall confine ourselves to *explicit* time stepping schemes in order to avoid the more costly implicit methods that require solving nonlinear systems of equations at each time step. Of course, with explicit methods there arises the issue of *stability* of the fully discrete schemes. We will not be exhaustive in our analysis but we will study as examples three simple, well known explicit Runge-Kutta temporal discretizations, that require, respectively, stability conditions of the type $k = O(h^2)$, $k = O(h^{4/3})$, and $k \leq \lambda_0 h$ for λ_0 sufficiently small, where k is the time step.

4.1. The explicit Euler scheme. Let M be a positive integer, $k = T/M$ denote the (uniform) time step, and put $t^n = nk$, $n = 0, 1, \dots, M$. We consider the standard Galerkin semidiscretizations with piecewise linear, continuous functions on a uniform spatial mesh on $[0, 1]$ with $h = 1/N$, given by the initial-value problems (2.7), (2.20) and (2.9), (2.20) in the case of the (CB) and the (SCB) systems, respectively. We discretize the systems in time with the explicit Euler scheme. Hence, we seek for $0 \leq n \leq M$ $H_h^n \in S_h^2$, $U_h^n \in S_{h,0}^2$, approximations of the solution $\eta(x, t^n)$, $u(x, t^n)$ of the (CB) system, such that for $0 \leq n \leq M - 1$

$$\begin{aligned} (H_h^{n+1} - H_h^n, \phi) + k(U_{hx}^n, \phi) + k((H_h^n U_h^n)_x, \phi) &= 0 \quad \forall \phi \in S_h^2, \\ a(U_h^{n+1} - U_h^n, \chi) + k(H_{hx}^n, \chi) + k(U_h^n U_{hx}^n, \chi) &= 0 \quad \forall \chi \in S_{h,0}^2, \end{aligned} \quad (4.1)$$

with

$$H_h^0 = I_h \eta^0, \quad U_h^0 = I_{h,0} u^0. \quad (4.2)$$

The analogous fully discrete approximation of the (SCB) system is defined, for $0 \leq n \leq M - 1$, by

$$\begin{aligned} (H_h^{n+1} - H_h^n, \phi) + k(U_{hx}^n, \phi) + \frac{k}{2}((H_h^n U_h^n)_x, \phi) &= 0 \quad \forall \phi \in S_h^2, \\ a(U_h^{n+1} - U_h^n, \chi) + k(H_{hx}^n, \chi) + \frac{3k}{2}(U_h^n U_{hx}^n, \chi) + \frac{k}{2}(H_h^n H_{hx}^n, \chi) &= 0 \quad \forall \chi \in S_{h,0}^2, \end{aligned} \quad (4.3)$$

with

$$H_h^0 = I_h \eta^0, \quad U_h^0 = I_{h,0} u^0. \quad (4.4)$$

Let $A : L^2 \rightarrow S_{h,0}^2$, be defined for $f \in L^2$ by

$$a(Af, \chi) = (f, \chi) \quad \forall \chi \in S_{h,0}^2, \quad (4.5)$$

i.e. as the discrete solution operator such that $w_h = Af$, where w_h is the standard Galerkin approximation in $S_{h,0}^2$ of the solution of the two-point bvp $-\frac{1}{3}w'' + w = f$, $0 \leq x \leq 1$, $w(0) = w(1) = 0$. From (4.5) we have immediately that

$$\|Af\|_1 \leq C\|f\|_{-1}, \quad (4.6)$$

where the $\|\cdot\|_{-1}$ norm is defined for $f \in L^2$ by

$$\|f\|_{-1} = \sup_{\substack{g \in H_0^1 \\ g \neq 0}} \frac{(f, g)}{\|g\|_1}.$$

With this notation in place and letting as before P denote the L^2 projection operator onto S_h^2 , we may rewrite the fully discrete scheme (4.1), (4.2) for (CB) as

$$\begin{aligned} H_h^{n+1} - H_h^n + kPU_{hx}^n + kP(H_h^n U_h^n)_x &= 0, \\ U_h^{n+1} - U_h^n + kAH_{hx}^n + kA(U_h^n U_{hx}^n) &= 0, \end{aligned} \quad (4.7)$$

for $0 \leq n \leq M-1$, with $H_h^0 = I_h \eta^0$, $U_h^0 = I_{h,0} u^0$. Similarly, for (SCB) we have from (4.3), (4.4)

$$\begin{aligned} H_h^{n+1} - H_h^n + kPU_{hx}^n + \frac{k}{2}P(H_h^n U_h^n)_x &= 0, \\ U_h^{n+1} - U_h^n + kAH_{hx}^n + \frac{3k}{2}A(U_h^n U_{hx}^n) + \frac{k}{2}A(H_h^n H_{hx}^n) &= 0, \end{aligned} \quad (4.8)$$

for $0 \leq n \leq M-1$, with $H_h^0 = I_h \eta^0$, $U_h^0 = I_{h,0} u^0$. We will prove error estimates for the schemes (4.7) and (4.8) by comparing H_h^n with $I_h \eta(t^n)$ and U_h^n with $I_{h,0} u(t^n)$. For this purpose, it is useful to establish the following estimates of the *truncation errors* of the interpolants. (In the sequel we shall analyze mainly the approximation of the (SCB) system; the analogous results for (CB) follow as in Sections 2 and 3. Frequently, we shall suppress the x variable, denoting e.g. $\eta(\cdot, t)$ by $\eta(t)$ etc.)

Lemma 4.1. *Suppose that the solution (η, u) of (SCB) is sufficiently smooth in $[0, T]$. Let $H(t) = I_h \eta(t)$, $U(t) = I_{h,0} u(t)$, and define $\psi(t) \in S_h^2$, $\zeta(t) \in S_{h,0}^2$ for $0 \leq t \leq T$ by*

$$H_t + PU_x + \frac{1}{2}P(HU)_x = \psi, \quad (4.9)$$

$$U_t + AH_x + \frac{3}{2}A(UU_x) + \frac{1}{2}A(HH_x) = \zeta. \quad (4.10)$$

Then

$$\begin{aligned} \|\psi\| &\leq Ch^{3/2}, \quad \|\psi_t\| \leq Ch^{3/2}, \\ \|\zeta\|_1 &\leq Ch^2, \quad \|\zeta_t\|_1 \leq Ch^2, \end{aligned} \quad (4.11)$$

hold for $0 \leq t \leq T$. An analogous result holds for (CB).

Proof. Subtracting the equations $P(\eta_t + u_x + \frac{1}{2}(u\eta)_x) = 0$ and (4.9), and putting $\rho := \eta - I_h \eta$, $\sigma := u - I_{h,0} u$, we obtain

$$P(\rho_t + [(1 + \frac{1}{2}\eta)\sigma]_x + \frac{1}{2}(u\rho)_x - \frac{1}{2}(\rho\sigma)_x) = -\psi.$$

Therefore, using the approximation properties of S_h^2 and $S_{h,0}^2$, and Lemma 2.2, we have

$$\|\psi\| \leq \|\rho_t\| + \|P[(1 + \frac{1}{2}\eta)\sigma]_x\| + \frac{1}{2}\|P(u\rho)_x\| + \|(\rho\sigma)_x\| \leq C(h^2 + h^{3/2} + h^2 + h^3) \leq Ch^{3/2}.$$

Similarly, since e.g. by Lemma 2.2

$$\|P[(1 + \frac{1}{2}\eta)\sigma]_{xt}\| \leq \|P(\frac{1}{2}\eta_t\sigma)_x\| + \|P[(1 + \frac{1}{2}\eta)\sigma_t]_x\| \leq Ch^{3/2},$$

we have

$$\|\psi_t\| \leq \|\rho_t\| + \|P[(1 + \frac{1}{2}\eta)\sigma]_{xt}\| + \frac{1}{2}\|P(u\rho)_{xt}\| + \|(\rho\sigma)_{xt}\| \leq C(h^2 + h^{3/2} + h^2 + h^3) \leq Ch^{3/2}.$$

Note now that for any $\chi \in S_{h,0}^2$, (4.5) and the fact that $(\sigma'_t, \chi') = 0$ yield

$$a(A(u_t - \frac{1}{3}u_{txx}) - U_t, \chi) = (u_t, \chi) + \frac{1}{3}(u_{tx}, \chi_x) - a(U_t, \chi) = a(\sigma_t, \chi) = (\sigma_t, \chi) = a(A\sigma_t, \chi).$$

Hence

$$A(u_t - \frac{1}{3}u_{txx}) - U_t = A\sigma_t,$$

which implies, in view of the second p.d.e. of (SCB) that

$$A\sigma_t + U_t + A(\eta_x + \frac{3}{2}uu_x + \frac{1}{2}\eta\eta_x) = 0.$$

Subtracting now this equation from (4.10) we see, after some algebra, that

$$A(\sigma_t + \rho_x + \frac{3}{2}[(u\sigma)_x - \sigma\sigma_x] + \frac{1}{2}[(\eta\rho)_x - \rho\rho_x]) = -\zeta. \quad (4.12)$$

Therefore, using (4.6) and the approximation properties of S_h^2 and $S_{h,0}^2$, we obtain

$$\begin{aligned} \|\zeta\|_1 &\leq C(\|\sigma_t\|_{-1} + \|\rho_x\|_{-1} + \|(u\sigma - \frac{1}{2}\sigma^2)_x\|_{-1} + \|(\eta\rho - \frac{1}{2}\rho^2)_x\|_{-1}) \\ &\leq C(\|\sigma_t\| + \|\rho\| + \|u\sigma\| + \|\sigma^2\| + \|\eta\rho\| + \|\rho^2\|) \leq Ch^2. \end{aligned}$$

Similarly, after differentiating (4.12) with respect to t , we see that

$$\|\zeta_t\|_1 \leq Ch^2,$$

thus ending the proof. The same results hold for (CB) of course. \square

We now proceed to prove error estimates for the explicit Euler-Galerkin schemes (4.7) and (4.8). We begin by a consistency result.

Lemma 4.2. *Suppose that the solution (η, u) of (SCB) is sufficiently smooth. Let $H^n := H(t^n) = I_h\eta(t^n)$, $U^n = U(t^n) = I_{h,0}u(t^n)$, and define, for $0 \leq n \leq M-1$, δ_1^n and δ_2^n by the equations*

$$\begin{aligned} \delta_1^n &:= H^{n+1} - H^n + kPU_x^n + \frac{k}{2}P(H^n U^n)_x, \\ \delta_2^n &:= U^{n+1} - U^n + kAH_x^n + \frac{3k}{2}A(U^n U_x^n) + \frac{k}{2}A(H^n H_x^n). \end{aligned}$$

Then

$$\max_{0 \leq n \leq M-1} (\|\delta_1^n\| + \|\delta_2^n\|_1) \leq Ck(k + h^{3/2}).$$

The analogous result holds for (CB) as well.

Proof. Using (4.9) and (4.10) we have, with $\psi^n = \psi(t^n)$, $\zeta^n = \zeta(t^n)$, that $\delta_1^n = H^{n+1} - H^n - kH_t^n + k\psi^n$, $\delta_2^n = U^{n+1} - U^n - kU_t^n + k\zeta^n$. Hence, for $0 \leq n \leq M-1$,

$$\begin{aligned} \|\delta_1^n\| + \|\delta_2^n\|_1 &\leq \|H^{n+1} - H^n - kH_t^n\| + k\|\psi^n\| + \|U^{n+1} - U^n - kU_t^n\|_1 + k\|\zeta^n\|_1 \\ &\leq C(k^2 + kh^{3/2} + k^2 + kh^2) \leq Ck(k + h^{3/2}), \end{aligned}$$

by Taylor's theorem and (4.11). \square

Proposition 4.1. *Suppose that the solutions (η, u) of (SCB) and (CB) are sufficiently smooth on $[0, T]$. Then, if $\mu = k/h^2$, there is a constant $C = C(\mu)$, which is an increasing continuous function of μ , such that*

$$\max_{0 \leq n \leq M} \|H_h^n - \eta(t^n)\| \leq C(k + h^{3/2}), \quad \max_{0 \leq n \leq M} \|U_h^n - u(t^n)\|_1 \leq C(k + h), \quad (4.13)$$

where (H_h^n, U_h^n) satisfy (4.7) or (4.8) as the case may be.

Proof. Consider the case of (SCB). We use the notation of Lemmas 4.1 and 4.2 and put $\varepsilon^n = H^n - H_h^n$, $e^n = U^n - U_h^n$. Using the definition of δ_1^n , δ_2^n , and (4.8) we obtain, after some straightforward computations, that for $0 \leq n \leq M-1$

$$\varepsilon^{n+1} = \varepsilon^n - kP[e_x^n + \frac{1}{2}(H^n e^n)_x - \frac{1}{2}(\varepsilon^n e^n)_x + \frac{1}{2}(U^n \varepsilon^n)_x] + \delta_1^n, \quad (4.14)$$

$$e^{n+1} = e^n - kA[\varepsilon_x^n + \frac{3}{2}(U^n e^n)_x - \frac{3}{2}(e^n e_x^n) + \frac{1}{2}(H^n \varepsilon^n)_x - \frac{1}{2}\varepsilon^n \varepsilon_x^n] + \delta_2^n. \quad (4.15)$$

It follows from (4.6) that

$$\|A(U^n e^n)_x\|_1 \leq C\|U^n e^n\| \leq C\|e^n\|, \quad (4.16)$$

$$\|A(H^n \varepsilon^n)_x\|_1 \leq C\|H^n \varepsilon^n\| \leq C\|\varepsilon^n\|. \quad (4.17)$$

Let now $0 \leq n^* \leq M-1$ be the maximal integer such that

$$\|\varepsilon^n\|_1 + \|e^n\|_1 \leq 1, \quad 0 \leq n \leq n^*. \quad (4.18)$$

Then, for $0 \leq n \leq n^*$, using (4.6), we have

$$\begin{aligned} \|P(e_x^n + \frac{1}{2}(H^n e^n)_x - \frac{1}{2}(\varepsilon^n e^n)_x)\| &\leq \|e_x^n\| + \frac{1}{2}\|(H^n e^n)_x\| + \frac{1}{2}\|(\varepsilon^n e^n)_x\| \\ &\leq C(\|e^n\|_1 + \|e^n\|_1 \|\varepsilon^n\|_1) \leq C\|e^n\|_1, \end{aligned} \quad (4.19)$$

$$\|A(e^n e_x^n)\|_1 + \|A(\varepsilon^n \varepsilon_x^n)\|_1 \leq C(\|(e^n)^2\| + \|(\varepsilon^n)^2\|) \leq C(\|e^n\| + \|\varepsilon^n\|). \quad (4.20)$$

Using (4.16)-(4.20) in (4.14) and (4.15) we have, for $0 \leq n \leq n^*$

$$\|\varepsilon^{n+1}\| \leq \|\varepsilon^n - \frac{k}{2}P(U^n \varepsilon^n)_x\| + Ck\|e^n\|_1 + \|\delta_1^n\|, \quad (4.21)$$

$$\|e^{n+1}\|_1 \leq \|e^n\|_1 + Ck(\|\varepsilon^n\| + \|e^n\|) + \|\delta_2^n\|_1. \quad (4.22)$$

Now

$$\|\varepsilon^n - \frac{k}{2}P(U^n \varepsilon^n)_x\|^2 = \|\varepsilon^n\|^2 + \frac{k^2}{4}\|P(U^n \varepsilon^n)_x\|^2 - k(\varepsilon^n, P(U^n \varepsilon^n)_x). \quad (4.23)$$

But, by inverse assumptions

$$\|P(U^n \varepsilon^n)_x\| \leq \|(U^n \varepsilon^n)_x\| \leq \frac{C}{h}\|\varepsilon^n\|.$$

In addition,

$$|(\varepsilon^n, P(U^n \varepsilon^n)_x)| = |(\varepsilon^n, (U^n \varepsilon^n)_x)| = \frac{1}{2}|(U_x^n \varepsilon^n, \varepsilon^n)| \leq C\|\varepsilon^n\|^2.$$

Hence, (4.23) yields

$$\|\varepsilon^n - \frac{k}{2}P(U^n \varepsilon^n)_x\|^2 \leq \|\varepsilon^n\|^2 + C\mu k\|\varepsilon^n\|^2 + Ck\|\varepsilon^n\|^2,$$

i.e. that

$$\|\varepsilon^n - \frac{k}{2}P(U^n \varepsilon^n)_x\| \leq (1 + C_\mu k)\|\varepsilon^n\|,$$

where C_μ is a generic polynomial in μ of degree one. Therefore (4.21) becomes

$$\|\varepsilon^{n+1}\| \leq \|\varepsilon^n\| + C_\mu k(\|\varepsilon^n\| + \|e^n\|_1) + \|\delta_1^n\|. \quad (4.24)$$

Hence, by (4.22), (4.24), and Lemma 4.2 we obtain for $0 \leq n \leq n^*$

$$\|\varepsilon^{n+1}\| + \|e^{n+1}\|_1 \leq \|\varepsilon^n\| + \|e^n\|_1 + C_\mu k(\|\varepsilon^n\| + \|e^n\|_1) + Ck(k + h^{3/2}).$$

By Gronwall's Lemma we conclude that

$$\|\varepsilon^n\| + \|e^n\|_1 \leq C(\mu, T)(k + h^{3/2}), \quad 0 \leq n \leq n^* + 1, \quad (4.25)$$

where $C(\mu, T) = \exp((C_0 + C_1\mu)T)$. Taking h sufficiently small, we see from the maximality property (4.18) of n^* that we may take $n^* = M - 1$, and the conclusion of the proposition follows from (2.1). The case of (CB) is entirely similar. \square

Remark 4.1. The estimate (4.25) and Sobolev's inequality imply that $\|e^n\|_\infty = O(k + h^{3/2})$. Therefore, $\max_n \|u(t^n) - U_h^n\|_\infty = O(k + h^{3/2})$.

Remark 4.2. Consider the *linearized* problem (2.43). In this case, the analogous fully discrete scheme is

$$\begin{aligned} H_h^{n+1} - H_h^n + kPU_{hx}^n &= 0, \\ U_h^{n+1} - U_h^n + kAH_{hx}^n &= 0, \end{aligned}$$

for $0 \leq n \leq M - 1$, with $H_h^0 = I_h \eta^0$, $U_h^0 = I_{h,0} u^0$. Consequently, using the notation of the proof of Proposition 4.1, we now have for $0 \leq n \leq M - 1$ the error equations

$$\begin{aligned} \varepsilon^{n+1} &= \varepsilon^n - kPe_x^n + \delta_1^n, \\ e^{n+1} &= e^n - kAe_x^n + \delta_2^n, \end{aligned}$$

from which there easily follows the estimate

$$\|\varepsilon^n\| + \|e^n\|_1 \leq C(k + h^{3/2}), \quad 0 \leq n \leq M,$$

and the conclusions of the analog of Proposition 4.1, without the stability restriction $k = O(h^2)$. In other words, the linearized system *is not stiff*. This may also be verified by examining the spectrum of the spatial discretization operator of the semidiscrete linearized system: The latter may be written for $0 \leq t \leq T$ in the form

$$\begin{aligned} \eta_{ht} + L_h u_h &= 0, \\ M_h u_{ht} + \tilde{L}_h \eta_h &= 0, \end{aligned}$$

where the operators $L_h : S_{h,0}^2 \rightarrow S_h^2$, $\tilde{L}_h : S_h^2 \rightarrow S_{h,0}^2$, $M_h : S_{h,0}^2 \rightarrow S_{h,0}^2$ are defined by the equations

$$\begin{aligned} (L_h \psi, \phi) &= (\psi_x, \phi) \quad \forall \psi \in S_{h,0}^2, \phi \in S_h^2, \\ (\tilde{L}_h \phi, \psi) &= (\phi_x, \psi) \quad \forall \phi \in S_h^2, \psi \in S_{h,0}^2, \\ (M_h \psi, \chi) &= a(\psi, \chi) \quad \forall \psi, \chi \in S_{h,0}^2. \end{aligned}$$

Hence, the semidiscrete system may be written on $S_h^2 \times S_{h,0}^2$ as

$$A_h W_{ht} + B_h W_h = 0,$$

where $W_h = [\eta_h, u_h]^T$, and

$$A_h = \begin{bmatrix} I & 0 \\ 0 & M_h \end{bmatrix}, \quad B_h = \begin{bmatrix} 0 & L_h \\ \tilde{L}_h & 0 \end{bmatrix}.$$

Therefore, the spectrum of the spatial discretization operator coincides with that of the generalized eigenvalue problem

$$B_h V_h = -\lambda_h A_h V_h. \quad (4.26)$$

Denoting the eigenfunctions as $V_h = [H_h, U_h]^T$, where H_h, U_h are elements of $S_h^2, S_{h,0}^2$, respectively, regarded as vector spaces over \mathbb{C} , we have $L_h U_h = -\lambda_h H_h$, $\tilde{L}_h H_h = -\lambda_h M_h U_h$, from which

$$\begin{aligned} (L_h U_h, H_h) &= -\lambda_h \|H_h\|^2, \\ (\tilde{L}_h H_h, U_h) &= -\lambda_h (M_h U_h, U_h), \end{aligned} \quad (4.27)$$

where the L^2 inner product for complex valued functions is defined as $(f, g) = \int_0^1 f(x) \overline{g(x)} dx$. Now

$$(L_h U_h, H_h) = (U_{hx}, H_h) = -(U_h, H_{hx}) = -\overline{(\tilde{L}_h H_h, U_h)}.$$

Therefore, from (4.27)

$$\lambda_h = \frac{-2i \operatorname{Im}(U_{hx}, H_h)}{\|H_h\|^2 + a(U_h, U_h)}. \quad (4.28)$$

We conclude that the spectrum of (4.26) consists of purely imaginary eigenvalues (and is symmetric about the origin as $-\lambda_h$ is also an eigenvalue corresponding to the eigenvector \bar{V}_h). Moreover, it follows from (4.28) that

$$|\lambda_h| \leq \frac{2\|U_{hx}\|\|H_h\|}{\|H_h\|^2 + a(U_h, U_h)} \leq C,$$

i.e. that the spectrum is bounded by a constant C independent of h , implying that the linearized semidiscrete problem is not stiff.

As we saw in Proposition 4.1, when the nonlinear (CB) or (SCB) system is discretized by the explicit Euler-Galerkin method, the mesh condition $k = O(h^2)$ is sufficient for stability. In a numerical experiment, we solved the nonlinear, nonhomogeneous (CB) system using as exact solution $\eta(x, t) = \exp(2t)(\cos(\pi x) + x + 2)$, $u(x, t) = \exp(xt)(\sin(\pi x) + x^3 - x^2)$ for $x \in [0, 1]$, and discretizing by the explicit Euler-standard Galerkin method with piecewise linear functions and fixed $N = 1/h = 400$. When we integrated up to $T = 1$ using $k = h^2$, we obtained an L^2 error for η that was approximately equal to $2.2090(-4)$. The accuracy degenerated when we took $k = h^\alpha$ with decreasing $\alpha < 2$. For example, at the temporal instance closest to $T = 1$ the computations yielded the following magnitudes of the L^2 errors of η :

$k = h^{1.8}$	3.8839(-4)
$k = h^{1.6}$	1.1257(-3)
$k = h^{1.4}$	3.6917(-3)
$k = h^{1.2}$	overflow at about $t = 0.8$.

4.2. The improved Euler method. We next study the temporal discretization of the initial-value problems (2.7), (2.20) and (2.9), (2.20) by the explicit, second-order accurate ‘improved Euler’ scheme, that may be written in the case of the o.d.e. $y' = f(t, y)$ in the two-step form

$$\begin{aligned} y^{n,1} &= y^n + \frac{k}{2}f(t^n, y^n), \\ y^{n+1} &= y^n + kf(t^n + \frac{k}{2}, y^{n,1}). \end{aligned}$$

Using notation analogous to that established in the previous paragraph, in the case of (CB) we seek for $0 \leq n \leq M$ $H_h^n \in S_h^2$, $U_h^n \in S_{h,0}^2$, and for $0 \leq n \leq M-1$ $H_h^{n,1} \in S_h^2$, $U_h^{n,1} \in S_{h,0}^2$, such that

$$\begin{aligned} H_h^{n,1} - H_h^n + \frac{k}{2}PU_{hx}^n + \frac{k}{2}P(H_h^n U_h^n)_x &= 0, \\ U_h^{n,1} - U_h^n + \frac{k}{2}AH_{hx}^n + \frac{k}{2}A(U_h^n U_{hx}^n) &= 0, \\ H_h^{n+1} - H_h^n + kPU_{hx}^{n,1} + kP(H_h^{n,1} U_h^{n,1})_x &= 0, \\ U_h^{n+1} - U_h^n + kAH_{hx}^{n,1} + kA(U_h^{n,1} U_{hx}^{n,1}) &= 0, \end{aligned} \tag{4.29}$$

for $0 \leq n \leq M-1$, with $H_h^0 = I_h \eta^0$, $U_h^0 = I_{h,0} u^0$. In the case of (SCB) the analogous equations are

$$\begin{aligned} H_h^{n,1} - H_h^n + \frac{k}{2}PU_{hx}^n + \frac{k}{4}P(H_h^n U_h^n)_x &= 0, \\ U_h^{n,1} - U_h^n + \frac{k}{2}AH_{hx}^n + \frac{3k}{4}A(U_h^n U_{hx}^n) + \frac{k}{4}A(H_h^n H_{hx}^n) &= 0, \\ H_h^{n+1} - H_h^n + kPU_{hx}^{n,1} + \frac{k}{2}P(H_h^{n,1} U_h^{n,1})_x &= 0, \\ U_h^{n+1} - U_h^n + kAH_{hx}^{n,1} + \frac{3k}{2}A(U_h^{n,1} U_{hx}^{n,1}) + \frac{k}{2}A(H_h^{n,1} H_{hx}^{n,1}) &= 0, \end{aligned} \tag{4.30}$$

for $0 \leq n \leq M-1$, with $H_h^0 = I_h \eta^0$, $U_h^0 = I_{h,0} u^0$. In order to study the consistency and convergence of the schemes, we let again $H^n = H(t^n) = I_h \eta(t^n)$, $U^n = U(t^n) = I_{h,0} u(t^n)$, where (η, u) is the solution of (CB) or (SCB), and define, in the case of (SCB), $(H^{n,1}, U^{n,1}) \in S_h^2 \times S_{h,0}^2$ for $0 \leq n \leq M-1$ by the equations

$$\begin{aligned} H^{n,1} - H^n + \frac{k}{2}PU_x^n + \frac{k}{4}P(H^n U^n)_x &= 0, \\ U^{n,1} - U^n + \frac{k}{2}AH_x^n + \frac{3k}{4}A(U^n U_x^n) + \frac{k}{4}A(H^n H_x^n) &= 0. \end{aligned} \tag{4.31}$$

In the case of (CB) $H^{n,1}$, $U^{n,1}$ are defined analogously. Our consistency result is:

Lemma 4.3. *Suppose that the solution (η, u) of (SCB) is sufficiently smooth and let $\lambda = k/h$. Define, for $0 \leq n \leq M-1$, δ_1^n , δ_2^n by the equations*

$$\delta_1^n = H^{n+1} - H^n + kPU_x^{n,1} + \frac{k}{2}P(H^{n,1} U^{n,1})_x, \tag{4.32}$$

$$\delta_2^n = U^{n+1} - U^n + kAH_x^{n,1} + \frac{3k}{2}A(U^{n,1} U_x^{n,1}) + \frac{k}{2}A(H^{n,1} H_x^{n,1}). \tag{4.33}$$

Then, there exists a constant $C_1 = C_1(\lambda)$, which is a polynomial of λ of degree one, such that

$$\max_{0 \leq n \leq M-1} (\|\delta_1^n\| + \|\delta_2^n\|_1) \leq C_1 k(k^2 + h^{3/2}).$$

The analogous result holds for (CB) as well.

Proof. Let $0 \leq n \leq M-1$. By (4.31), (4.9) and (4.10) we have

$$H^{n,1} = H^n + \frac{k}{2}H_t^n - \frac{k}{2}\psi^n, \quad U^{n,1} = U^n + \frac{k}{2}U_t^n - \frac{k}{2}\zeta^n. \tag{4.34}$$

From these expressions, after some algebra, we obtain

$$H^{n,1} U^{n,1} = H^n U^n + \frac{k}{2}(HU)_t^n + w_1^n,$$

where

$$w_1^n := \frac{k^2}{4}H_t^n U_t^n - \frac{k}{2}(U^n + \frac{k}{2}U_t^n)\psi^n - \frac{k}{2}(H^n + \frac{k}{2}H_t^n)\zeta^n + \frac{k^2}{4}\psi^n \zeta^n. \tag{4.35}$$

Hence, by (4.32), (4.34), (4.9), and the above we obtain

$$\delta_1^n = H^{n+1} - H^n - kH_t^n - \frac{k^2}{2}H_{tt}^n + k\psi^n + \frac{k^2}{2}\psi_t^n - \frac{k^2}{2}P\zeta_x^n + \frac{k}{2}Pw_{1x}^n. \tag{4.36}$$

Now, (4.35), in view of (4.11) and the approximation and inverse properties of S_h^2 and $S_{h,0}^2$, gives

$$\begin{aligned}\|w_1^n\|_1 &\leq C(k^2\|H_t^n\|_1\|U_t^n\|_1 + k\|\psi^n\|_1(\|U^n\|_1 + k\|U_t^n\|_1) \\ &\quad + k\|\zeta^n\|_1(\|H^n\|_1 + k\|H_t^n\|_1) + k^2\|\psi^n\|_1\|\zeta^n\|_1) \\ &\leq c(k^2 + kh^{-1}h^{3/2}(1 + ck) + kh^2(1 + ck) + k^2h^{-1}h^{7/2}) \\ &\leq c(k^2 + \lambda h^{3/2}).\end{aligned}$$

Therefore, by Taylor's theorem and (4.11) we have

$$\|\delta_1^n\| \leq c(k^3 + kh^{3/2} + k^2h^{3/2} + k^2h^2 + k(k^2 + \lambda h^{3/2})) \leq C_1k(k^2 + h^{3/2}), \quad (4.37)$$

where C_1 is a constant that is a polynomial of λ of degree one. (Such constants will be generically denoted by C_1 in the sequel of this proof.) In order to estimate $\|\delta_2^n\|_1$ note that by (4.34)

$$U^{n,1}U_x^{n,1} = U^nU_x^n + \frac{k}{2}(UU_x)_t^n + w_2^n, \quad (4.38)$$

where

$$w_2^n := \frac{k^2}{4}U_t^nU_{tx}^n - \frac{k}{2}((U^n + \frac{k}{2}U_t^n)\zeta^n)_x + \frac{k^2}{4}\zeta^n\zeta_x^n.$$

By (4.11) and the approximation properties of $S_{h,0}^2$ we have

$$\|w_2^n\| \leq C(k^2 + kh^2). \quad (4.39)$$

Similarly,

$$H^{n,1}H_x^{n,1} = H^nH_x^n + \frac{k}{2}(HH_x)_t^n + w_3^n, \quad (4.40)$$

where

$$w_3^n := \frac{k^2}{4}H_t^nH_{tx}^n - \frac{k}{2}((H^n + \frac{k}{2}H_t^n)\psi^n)_x + \frac{k^2}{4}\psi^n\psi_x^n.$$

By (4.11) and the approximation and inverse properties of S_h^2 we have

$$\|w_3^n\| \leq C(k^2 + \lambda h^{3/2}). \quad (4.41)$$

By (4.33), (4.35), (4.38), and (4.40), we see now that

$$\delta_2^n = (U^{n+1} - U^n - kU_t^n - \frac{k^2}{2}U_{tt}^n) + k\zeta^n + \frac{k^2}{2}\zeta_t^n - \frac{k^2}{2}A\psi_x^n + \frac{3k}{2}Aw_2^n + \frac{k}{2}Aw_3^n.$$

Therefore, by Taylor's theorem, (4.11), (4.6), (4.39), (4.41),

$$\|\delta_2^n\|_1 \leq c(k^3 + kh^2 + k^2h^2 + k^2h^{3/2} + k^3 + k^2h^2 + k^3 + \lambda kh^{3/2}) \leq C_1k(k^2 + h^{3/2}),$$

which, with (4.37), concludes the proof of the Lemma. The case of (CB) is entirely analogous. \square

For the stability and convergence of the fully discrete scheme it does not suffice to suppose that $k = O(h)$. The following result shows that the stronger condition $k = O(h^{4/3})$ is sufficient.

Proposition 4.2. *Suppose that the solutions (η, u) of (SCB) and (CB) are sufficiently smooth on $[0, T]$. Then, if $\mu = k/h^{4/3}$, there is a constant $C = C(\mu)$, which is an increasing continuous function of μ , such that*

$$\max_{0 \leq n \leq M} \|H_h^n - \eta(t^n)\| \leq C(k^2 + h^{3/2}), \quad \max_{0 \leq n \leq M} \|U_h^n - u(t^n)\|_1 \leq C(k^2 + h).$$

Proof. We consider (SCB), and put $\varepsilon^n = H^n - H_h^n$, $e^n = U^n - U_h^n$, $\theta^n = H^{n,1} - H_h^{n,1}$, and $\xi^n = U^{n,1} - U_h^{n,1}$. We will show that

$$\max_{0 \leq n \leq M} (\|\varepsilon^n\| + \|e^n\|_1) \leq C(k^2 + h^{3/2}), \quad (4.42)$$

from which the conclusion of the proposition follows. From (4.30), (4.32), (4.33) we have, for $0 \leq n \leq M-1$

$$\varepsilon^{n+1} = \varepsilon^n - kP\xi_x^n - \frac{k}{2}P(H^{n,1}U^{n,1} - H_h^{n,1}U_h^{n,1})_x + \delta_1^n, \quad (4.43)$$

and

$$e^{n+1} = e^n - kA\theta_x^n - \frac{3k}{2}A(U^{n,1}U_x^{n,1} - U_h^{n,1}U_{hx}^{n,1}) - \frac{k}{2}A(H^{n,1}H_x^{n,1} - H_h^{n,1}H_{hx}^{n,1}) + \delta_2^n. \quad (4.44)$$

From Lemma 4.3 we have an estimate of $\|\delta_1^n\| + \|\delta_2^n\|_1$. Our goal is to obtain suitable estimates of the remaining terms of the right-hand sides of (4.43) and (4.44) in terms of $\|\varepsilon^n\| + \|e^n\|_1$. To do this, note first that (4.31) and (4.30) give

$$\theta^n = \varepsilon^n - \frac{k}{2}Pe_x^n - \frac{k}{4}P(H^nU^n - H_h^nU_h^n)_x.$$

But $H^n U^n - H_h^n U_h^n = H^n e^n - \varepsilon^n e^n + U^n \varepsilon^n$. Therefore

$$\theta^n = \varepsilon^n - \frac{k}{4} \rho_1^n - \frac{k}{2} \omega_1^n, \quad (4.45)$$

where

$$\rho_1^n := P(U^n \varepsilon^n)_x, \quad (4.46)$$

and

$$\omega_1^n := P e_x^n + \frac{1}{2} P(H^n e^n)_x - \frac{1}{2} P(\varepsilon^n e^n)_x. \quad (4.47)$$

Similarly,

$$\xi^n = e^n - \frac{k}{2} A \varepsilon_x^n - \frac{3k}{4} A(U^n U_x^n - U_h^n U_{hx}^n) - \frac{k}{4} A(H^n H_x^n - H_h^n H_{hx}^n).$$

But $U^n U_x^n - U_h^n U_{hx}^n = (U^n e^n)_x - e^n e_x^n$, $H^n H_x^n - H_h^n H_{hx}^n = (H^n \varepsilon^n)_x - \varepsilon^n \varepsilon_x^n$. Therefore

$$\xi^n = e^n - \frac{k}{2} \omega_2^n, \quad (4.48)$$

where

$$\omega_2^n := A \varepsilon_x^n + \frac{3}{2} A(U^n e^n)_x - \frac{3}{2} A(e^n e_x^n) + \frac{1}{2} A(H^n \varepsilon^n)_x - \frac{1}{2} A(\varepsilon^n \varepsilon_x^n). \quad (4.49)$$

In addition, by (4.45) we have

$$H^{n,1} U^{n,1} - H_h^{n,1} U_h^{n,1} = U^n \varepsilon^n - \frac{k}{4} U^n \rho_1^n - \frac{k}{2} U^n \omega_1^n + \omega_3^n, \quad (4.50)$$

where

$$\omega_3^n := (U^{n,1} - U^n) \theta^n + H^{n,1} \xi^n - \theta^n \xi^n. \quad (4.51)$$

From (4.43), (4.45)-(4.50) we conclude therefore that for $0 \leq n \leq M-1$

$$\varepsilon^{n+1} = \varepsilon^n - \frac{k}{2} \rho_1^n + \frac{k^2}{8} \rho_2^n - k \omega_4^n + \delta_1^n, \quad (4.52)$$

where

$$\rho_2^n := P(U^n \rho_1^n)_x, \quad (4.53)$$

and

$$\omega_4^n := P e_x^n - \frac{k}{2} P \omega_{2x}^n - \frac{k}{4} P(U^n \omega_1^n)_x + \frac{1}{2} P \omega_{3x}^n. \quad (4.54)$$

Finally, using the identities $U^{n,1} U_x^{n,1} - U_h^{n,1} U_{hx}^{n,1} = (U^{n,1} \xi^n)_x - \xi^n \xi_x^n$ and $H^{n,1} H_x^{n,1} - H_h^{n,1} H_{hx}^{n,1} = (H^{n,1} \theta^n)_x - \theta^n \theta_x^n$, we obtain from (4.44) that for $0 \leq n \leq M-1$

$$e^{n+1} = e^n - k A \theta_x^n - \frac{3k}{2} A((U^{n,1} \xi^n)_x - \xi^n \xi_x^n) - \frac{k}{2} A((H^{n,1} \theta^n)_x - \theta^n \theta_x^n) + \delta_2^n. \quad (4.55)$$

We now estimate the various terms in the right-hand sides of (4.52) and (4.55). Let $0 \leq n^* \leq M-1$ be the maximal index for which

$$\|\varepsilon^n\|_1 + \|e^n\|_1 \leq 1, \quad 0 \leq n \leq n^*. \quad (4.56)$$

Then, by (4.47), the approximation properties of S_h^2 and (4.56), we have, for $0 \leq n \leq n^*$

$$\|\omega_1^n\| \leq \|e^n\|_1 + C \|H^n\|_1 \|e^n\|_1 + C \|\varepsilon^n\|_1 \|e^n\|_1 \leq C \|e^n\|_1. \quad (4.57)$$

By (4.49), the approximation properties of S_h^2 , $S_{h,0}^2$, and (4.6) there follows for $0 \leq n \leq n^*$

$$\|\omega_2^n\|_1 \leq C(\|\varepsilon^n\| + \|e^n\| + \|e^n\|_1 \|e^n\| + \|\varepsilon^n\| + \|\varepsilon^n\|_1 \|\varepsilon^n\|) \leq C(\|\varepsilon^n\| + \|e^n\|). \quad (4.58)$$

Hence, by (4.48), for $0 \leq n \leq n^*$

$$\|\xi^n\|_1 \leq C(\|\varepsilon^n\| + \|e^n\|_1). \quad (4.59)$$

In addition, by (4.45), (4.46), (4.57) and the inverse assumptions we have for $0 \leq n \leq n^*$

$$\|\theta^n\| \leq \|\varepsilon^n\| + C \frac{k}{2} \|\varepsilon^n\| + C \|e^n\|_1 \leq (1 + C\lambda) \|\varepsilon^n\| + C \|e^n\|_1 \leq C_\lambda (\|\varepsilon^n\| + \|e^n\|_1), \quad (4.60)$$

where we have put again $\lambda = k/h$; in the sequel, C_λ will denote various constants that depend polynomially on λ . Note also that in view of (4.56) we have for $0 \leq n \leq n^*$, from (4.57), (4.46) and inverse inequalities

$$\|\theta^n\|_1 \leq \|\varepsilon^n\|_1 + Ck \|\rho_1^n\|_1 + Ck \|\omega_1^n\|_1 \leq \|\varepsilon^n\|_1 + C\lambda \|\rho_1^n\| + C\lambda \|\omega_1^n\| \leq C_\lambda. \quad (4.61)$$

By (4.51), (4.34), (4.11), (4.60), (4.59) and (4.56) we have for $0 \leq n \leq n^*$

$$\begin{aligned} \|\omega_3^n\| &\leq C(\|U^{n,1} - U^n\|_1 \|\theta^n\| + \|H^{n,1}\| \|\xi^n\|_1 + \|\theta^n\| \|\xi^n\|_1) \\ &\leq C((k + h^2) \|\theta^n\| + \|\xi^n\|_1 + \|\theta^n\| \|\xi^n\|_1) \\ &\leq C_\lambda (\|\varepsilon^n\| + \|e^n\|_1). \end{aligned} \quad (4.62)$$

Also, by (4.60), (4.59), (4.61), for $0 \leq n \leq n^*$

$$\begin{aligned} \|\omega_3^n\|_1 &\leq C(\|U^{n,1} - U^n\|_1 \|\theta^n\|_1 + \|H^{n,1}\|_1 \|\xi^n\|_1 + \|\theta^n\|_1 \|\xi^n\|_1) \\ &\leq (k + h^2)h^{-1}C_\lambda(\|\varepsilon^n\| + \|e^n\|_1) + C_\lambda(\|\varepsilon^n\| + \|e^n\|_1) + C_\lambda(\|\varepsilon^n\| + \|e^n\|_1) \\ &\leq C_\lambda(\|\varepsilon^n\| + \|e^n\|_1). \end{aligned} \quad (4.63)$$

Hence, by (4.54), (4.58), (4.57), (4.62) and the inverse inequalities we have for $0 \leq n \leq n^*$

$$\|\omega_4^n\| \leq \|e^n\|_1 + Ck(\|\varepsilon^n\| + \|e^n\|) + C_\lambda\|e^n\|_1 + C_\lambda(\|\varepsilon^n\| + \|e^n\|_1) \leq C_\lambda(\|\varepsilon^n\| + \|e^n\|_1).$$

Therefore, in the right-hand side of (4.52) we have for $0 \leq n \leq n^*$ in view of Lemma 4.3,

$$\| -k\omega_4^n + \delta_1^n \| \leq C_\lambda k(\|\varepsilon^n\| + \|e^n\|_1) + C_\lambda k(k + h^{3/2}). \quad (4.64)$$

We embark now upon obtaining a sharp L^2 -estimate of the remaining term $\varepsilon^n - \frac{k}{2}\rho_1^n + \frac{k^2}{8}\rho_2^n$ in (4.52). We have

$$\|\varepsilon^n - \frac{k}{2}\rho_1^n + \frac{k^2}{8}\rho_2^n\|^2 = \|\varepsilon^n\|^2 + \frac{k^2}{4}\|\rho_1^n\|^2 + \frac{k^4}{64}\|\rho_2^n\|^2 - k(\varepsilon^n, \rho_1^n) + \frac{k^2}{4}(\varepsilon^n, \rho_2^n) - \frac{k^3}{8}(\rho_1^n, \rho_2^n).$$

Now, by (4.46)

$$(\varepsilon^n, \rho_1^n) = (\varepsilon^n, (U^n \varepsilon^n)_x) = \frac{1}{2}(U_x^n \varepsilon^n, \varepsilon^n).$$

Also, by (4.53), (4.46)

$$\begin{aligned} (\varepsilon^n, \rho_2^n) &= -(\varepsilon_x^n, U^n \rho_1^n) = -\|\rho_1^n\|^2 + (U_x^n \varepsilon^n, \rho_1^n), \\ (\rho_1^n, \rho_2^n) &= \frac{1}{2}(U_x^n \rho_1^n, \rho_1^n). \end{aligned}$$

We conclude that

$$\|\varepsilon^n - \frac{k}{2}\rho_1^n + \frac{k^2}{8}\rho_2^n\|^2 = \|\varepsilon^n\|^2 + \frac{k^4}{64}\|\rho_2^n\|^2 - \frac{k}{2}(U_x^n \varepsilon^n, \varepsilon^n) + \frac{k^2}{4}(U_x^n \varepsilon^n, \rho_1^n) - \frac{k^3}{16}(U_x^n \rho_1^n, \rho_1^n). \quad (4.65)$$

Now, by the approximation and inverse properties of $S_{h,0}^2$ we have by (4.46), (4.53)

$$\begin{aligned} |(U_x^n \varepsilon^n, \varepsilon^n)| &\leq C\|\varepsilon^n\|^2, \\ |(U_x^n \varepsilon^n, \rho_1^n)| &\leq C\|\varepsilon^n\| \|\rho_1^n\| \leq Ch^{-1}\|\varepsilon^n\|^2, \\ |(U_x^n \rho_1^n, \rho_1^n)| &\leq C\|\rho_1^n\|^2 \leq Ch^{-2}\|\varepsilon^n\|^2, \\ \|\rho_2^n\| &\leq Ch^{-2}\|\varepsilon^n\|. \end{aligned}$$

Inserting these estimates in (4.65) and recalling that $\mu = k/h^{4/3}$ we are led to the inequality

$$\|\varepsilon^n - \frac{k}{2}\rho_1^n + \frac{k^2}{8}\rho_2^n\|^2 \leq (1 + Ck\mu^3 + Ck + Ck\lambda + Ck\lambda^2)\|\varepsilon^n\|^2 \leq (1 + C_\mu k)\|\varepsilon^n\|^2,$$

and, hence,

$$\|\varepsilon^n - \frac{k}{2}\rho_1^n + \frac{k^2}{8}\rho_2^n\| \leq (1 + C_\mu k)\|\varepsilon^n\|, \quad (4.66)$$

where, by C_μ we denote a constant depending polynomially on μ ; we have replaced C_λ 's by C_μ 's since $\lambda = h^{1/3}\mu \leq \mu$. We finally obtain from (4.66), (4.64) and (4.52), for $0 \leq n \leq n^*$, that

$$\|\varepsilon^{n+1}\| \leq \|\varepsilon^n\| + C_\mu k(\|\varepsilon^n\| + \|e^n\|_1) + C_\mu k(k^2 + h^{3/2}). \quad (4.67)$$

We now estimate the $\|\cdot\|_1$ norm of the various terms in the right-hand side of (4.55). For $0 \leq n \leq n^*$, by (4.6), (4.34), (4.56), and (4.59) we have

$$\|A((U^{n,1}\xi^n)_x - \xi^n \xi_x^n)\|_1 \leq C\|U^{n,1}\xi^n - \frac{1}{2}(\xi^n)^2\| \leq C\|\xi^n\|_1 \leq C(\|\varepsilon^n\| + \|e^n\|_1).$$

Similarly, for $0 \leq n \leq n^*$, by (4.60), we have

$$\|A((H^{n,1}\theta^n)_x - \theta^n \theta_x^n)\|_1 \leq C\|H^{n,1}\theta^n - \frac{1}{2}(\theta^n)^2\| \leq C_\lambda\|\theta^n\| \leq C_\lambda(\|\varepsilon^n\| + \|e^n\|_1).$$

Therefore, using (4.55), (4.60) and Lemma 4.3 we have for $0 \leq n \leq n^*$

$$\begin{aligned} \|e^{n+1}\|_1 &\leq \|e^n\|_1 + Ck\|\theta^n\| + C_\lambda k(\|\varepsilon^n\| + \|e^n\|_1) + C_\lambda k(k^2 + h^{3/2}) \\ &\leq \|e^n\|_1 + C_\lambda k(\|\varepsilon^n\| + \|e^n\|_1) + C_\lambda k(k^2 + h^{3/2}). \end{aligned} \quad (4.68)$$

Adding now (4.67) and (4.68), we conclude for $0 \leq n \leq n^*$ that

$$\|\varepsilon^{n+1}\| + \|e^{n+1}\|_1 \leq (1 + C_\mu k)(\|\varepsilon^n\| + \|e^n\|_1) + C_\mu k(k^2 + h^{3/2}).$$

Using Gronwall's Lemma and taking h sufficiently small we conclude that n^* may be taken equal to $M - 1$ and there holds that

$$\|\varepsilon^n\| + \|e^n\|_1 \leq \exp(C_\mu T)(k^2 + h^{3/2}), \quad 0 \leq n \leq M,$$

i.e. that (4.42) is valid; the conclusion of the proposition follows. \square

Remark 4.3. Arguing as in Remark 4.1 we have again $\|U_h^n - u(t^n)\|_\infty = O(k^2 + h^{3/2})$.

Remark 4.4. If we repeat the numerical experiment in Remark 4.2 using the improved Euler method for time stepping, we obtain, for $N = 1/h = 400$, the evolution of the L^2 errors $\|H_h^n - \eta(t^n)\|$ when $k = h$ and $k = h^{4/3}$ shown in Table 4.1. These results suggest that the condition $k = h^{4/3}$ is probably necessary as well for the stability of the scheme. If we repeat the experiment using $k = h$ and the fourth-order explicit classical R-K scheme for time stepping (this scheme, as will be shown in the next paragraph, is stable when $k = h$.) we obtain errors approximately equal to those of the last column of Table 4.1: For example at $t^n = 0.95$ and $t^n = 1.0$ the analogous errors are equal to $0.1810(-3)$ and $0.1954(-3)$, respectively. This implies that the errors of the last column of Table 4.1 are essentially due to the spatial discretization.

t^n	$k = h$	t^n	$k = h^{4/3}$
0.05	0.9594(-5)	0.05090	0.9833(-5)
0.1	0.1857(-4)	0.10179	0.1890(-4)
0.3	0.5436(-4)	0.30198	0.5394(-4)
0.5	0.8538(-4)	0.50217	0.8393(-4)
0.7	0.1211(-3)	0.70236	0.1206(-3)
0.8	0.3949(-3)	0.80075	0.1448(-3)
0.825	0.2214(-2)	0.82450	0.1507(-3)
0.85	0.1445(-1)	0.85165	0.1572(-3)
0.9	0.8082	0.90254	0.1691(-3)
0.95	0.7706(+18)	0.95005	0.1811(-3)
1.0	<i>overflow</i>	1.00026	0.1963(-3)

TABLE 4.1. L^2 -errors $\|H_h^n - \eta(t^n)\|$ for the standard Galerkin method with piecewise linear continuous functions with $h = 1/400$ and improved Euler time stepping with $k = h$ and $k = h^{4/3}$; example of Remark 4.2.

4.3. Fourth-order Runge-Kutta scheme with cubic splines. Our third example is time stepping with the classical, fourth-order accurate four-stage explicit Runge-Kutta scheme, written in the case of the o.d.e. $y' = f(t, y)$ in the three-step form

$$\begin{aligned} y^{n,1} &= y^n + \frac{k}{2} f(t^n + \frac{k}{2}, y^n), \\ y^{n,2} &= y^n + \frac{k}{2} f(t^n + \frac{k}{2}, y^{n,1}), \\ y^{n,3} &= y^n + k f(t^n + k, y^{n,2}), \\ y^{n+1} &= y^n + k \left(\frac{1}{6} f(t^n, y^n) + \frac{1}{3} f(t^n + \frac{k}{2}, y^{n,1}) + \frac{1}{3} f(t^n + \frac{k}{2}, y^{n,2}) + \frac{1}{6} f(t^n + k, y^{n,3}) \right). \end{aligned}$$

We will couple this scheme with a space discretization that uses cubic splines on a uniform mesh on $[0, 1]$. We recall that the semidiscrete problem can be written, e.g. in the case of the (SCB), in the form (3.8) or, equivalently, as follows. We seek $(\eta_h, u_h) \in C^1(0, T; S_h^4 \times S_{h,0}^4)$ such that

$$\begin{aligned} \eta_{ht} + P u_{hx} + \frac{1}{2} P(\eta_h u_h)_x &= 0, \\ u_{ht} + A \eta_{hx} + \frac{3}{2} A(u_h u_{hx}) + \frac{1}{2} A(\eta_h \eta_{hx}) &= 0, \\ \eta_h(0) = I_h \eta_0, \quad u_h(0) &= R_h u_0, \end{aligned} \quad 0 \leq t \leq T, \quad (4.69)$$

where $P : L^2 \rightarrow S_h^4$ is the L^2 -projection operator onto S_h^4 , $A : L^2 \rightarrow S_{h,0}^4$ is defined as in (4.5), but now in $S_{h,0}^4$, I_h is the interpolant in S_h^4 , and R_h is introduced in Paragraph 3.1. The semidiscrete scheme for the (CB) system is defined analogously.

We start with an estimation of the truncation errors of the semidiscrete equations when applied to $I_h \eta(t)$ and $R_h u(t)$ similar to that of Lemma 4.1.

Lemma 4.4. Suppose that the solution (η_h, u_h) of (SCB) is sufficiently smooth in $[0, T]$. Let $H(t) = I_h \eta(t)$, $U(t) = R_h u(t)$ and define $\psi = \psi(t) \in S_h^4$, $\zeta = \zeta(t) \in S_{h,0}^4$, for $0 \leq t \leq T$ by

$$H_t + PU_x + \frac{1}{2}P(HU)_x = \psi, \quad (4.70)$$

$$U_t + AH_x + \frac{3}{2}A(UU_x) + \frac{1}{2}A(HH_x) = \zeta. \quad (4.71)$$

Then, for $j = 0, 1, 2, 3$

$$\|\partial_t^j \psi\| \leq Ch^{3.5} \sqrt{\ln 1/h}, \quad \|\partial_t^j \zeta\|_1 \leq Ch^4, \quad (4.72)$$

hold for $0 \leq t \leq T$. An analogous result holds for (CB) as well.

Proof. Subtracting the equations

$$P\eta_t + Pu_x + \frac{1}{2}P(\eta u)_x = 0,$$

$$H_t + PU_x + \frac{1}{2}P(HU)_x = \psi,$$

we obtain, putting $\rho := \eta - H$, $\sigma := u - U$

$$P\rho_t + P\sigma_x + \frac{1}{2}P(\eta u - HU)_x = -\psi.$$

Since

$$\eta u - HU = \eta(u - U) + U(\eta - H) = \eta(u - U) - (u - U)(\eta - H) + u(\eta - H) = \eta\sigma + u\rho - \rho\sigma,$$

we have

$$P\rho_t + P\sigma_x + \frac{1}{2}P(\eta\sigma)_x + \frac{1}{2}P(u\rho)_x - \frac{1}{2}P(\rho\sigma)_x = -\psi,$$

and conclude from Lemmas 3.2 and 3.4 that

$$\|\partial_t^j \psi\| \leq Ch^{3.5} \sqrt{\ln 1/h}, \quad j = 0, 1, 2, 3,$$

which is the first estimate in (4.72) By the definition of A we have

$$A(u_t - \frac{1}{3}u_{txx}) = U_t.$$

Subtracting now the equations

$$A(u_t - \frac{1}{3}u_{txx}) + A\eta_x + \frac{3}{2}A(uu_x) + \frac{1}{2}A(\eta\eta_x) = 0,$$

$$U_t + AH_x + \frac{3}{2}A(UU_x) + \frac{1}{2}A(HH_x) = \zeta,$$

we obtain

$$A\rho_x + \frac{3}{2}A(uu_x - UU_x) + \frac{1}{2}A(\eta\eta_x - HH_x) = -\zeta.$$

But

$$uu_x - UU_x = (u\sigma)_x - \sigma\sigma_x, \quad \eta\eta_x - HH_x = (\eta\rho)_x - \rho\rho_x.$$

Hence

$$A\rho_x + \frac{3}{2}(A(u\sigma)_x - A(\sigma\sigma_x)) + \frac{1}{2}(A(\eta\rho)_x - A(\rho\rho_x)) = -\zeta,$$

and from (4.6) and the approximation properties of S_h^4 , $S_{h,0}^4$ we conclude

$$\|\partial_t^j \zeta\|_1 \leq Ch^4,$$

for $j = 0, 1, 2, 3$, thus proving the Lemma. The (CB) case is entirely similar. \square

We now define the fully discrete scheme. We consider only the case of (SCB), that of (CB) being analogous. We let as usual M be a positive integer, $k = T/M$ and $t^n = nk$, for $n = 0, 1, \dots, M$. For $0 \leq n \leq M$ we seek $(H_h^n, U_h^n) \in S_h^4 \times S_{h,0}^4$ approximations of $\eta(t^n)$, $u(t^n)$, and for $0 \leq n \leq M-1$ $(H_h^{n,j}, U_h^{n,j}) \in S_h^4 \times S_{h,0}^4$, $j = 1, 2, 3$, such that for $0 \leq n \leq M-1$

$$\begin{aligned} H_h^{n,j} - H_h^n + ka_j P(U_h^{n,j-1} + \frac{1}{2}H_h^{n,j-1}U_h^{n,j-1})_x &= 0, \\ U_h^{n,j} - U_h^n + ka_j A(H_{hx}^{n,j-1} + \frac{3}{2}U_h^{n,j-1}U_{hx}^{n,j-1} + \frac{1}{2}H_h^{n,j-1}H_{hx}^{n,j-1}) &= 0, \end{aligned} \quad (4.73)$$

for $j = 1, 2, 3$ and

$$\begin{aligned} H_h^{n+1} - H_h^n + kP \left[\sum_{j=1}^4 b_j (U_h^{n,j-1} + \frac{1}{2} H_h^{n,j-1} U_h^{n,j-1}) \right]_x &= 0, \\ U_h^{n+1} - U_h^n + kA \left[\sum_{j=1}^4 b_j (H_{hx}^{n,j-1} + \frac{3}{2} U_{hx}^{n,j-1} U_h^{n,j-1} + \frac{1}{2} H_{hx}^{n,j-1} H_h^{n,j-1}) \right] &= 0, \end{aligned} \quad (4.74)$$

where

$$H_h^{n,0} = H_h^n, \quad U_h^{n,0} = U_h^n, \quad a_1 = a_2 = 1/2, \quad a_3 = 1, \quad b_1 = b_4 = 1/6, \quad b_2 = b_3 = 1/3,$$

and

$$H_h^0 = \eta_h(0), \quad U_h^0 = u_h(0).$$

We first investigate the consistency of this scheme. We define H, U as in Lemma 4.4 and put $H^n = H(t^n)$, $U^n = U(t^n)$. We let $V^{n,j} \in S_h^4$, $W^{n,j} \in S_{h,0}^4$, for $j = 0, 1, 2, 3$ be defined for $0 \leq n \leq M-1$ by the equations

$$\begin{aligned} V^{n,j} - H^n + ka_j P (W^{n,j-1} + \frac{1}{2} V^{n,j-1} W^{n,j-1})_x &= 0, \\ W^{n,j} - U^n + ka_j A (V_x^{n,j-1} + \frac{3}{2} W^{n,j-1} W_x^{n,j-1} + \frac{1}{2} V^{n,j-1} V_x^{n,j-1}) &= 0, \end{aligned} \quad (4.75)$$

and

$$V^{n,0} = H^n, \quad W^{n,0} = U^n.$$

Before proving our consistency estimates we introduce some notation that will simplify the computations. Let

$$\Phi = U + \frac{1}{2} HU, \quad F = H_x + \frac{3}{2} UU_x + \frac{1}{2} HH_x, \quad \Phi^n = \Phi(t^n), \quad F^n = F(t^n),$$

and consider the functions

$$\begin{aligned} \Phi_h^{n,j} &= U_h^{n,j} + \frac{1}{2} H_h^{n,j} U_h^{n,j}, \\ F_h^{n,j} &= H_{hx}^{n,j} + \frac{3}{2} U_h^{n,j} U_{hx}^{n,j} + \frac{1}{2} H_h^{n,j} H_{hx}^{n,j}, \end{aligned}$$

for $j = 0, 1, 2, 3$. With a slight abuse of notation we put for $j = 0, 1, 2, 3$

$$\begin{aligned} \Phi^{n,j} &= W^{n,j} + \frac{1}{2} V^{n,j} W^{n,j}, \\ F^{n,j} &= V_x^{n,j} + \frac{3}{2} W^{n,j} W_x^{n,j} + \frac{1}{2} V^{n,j} V_x^{n,j}. \end{aligned}$$

We may then write (4.70) and (4.71) as

$$H_t + P\Phi_x = \psi, \quad (4.76)$$

$$U_t + AF = \zeta, \quad (4.77)$$

and (4.73), (4.74), respectively, as

$$\begin{aligned} H_h^{n,j} - H_h^n + ka_j P \Phi_{hx}^{n,j-1} &= 0, \\ U_h^{n,j} - U_h^n + ka_j A F_h^{n,j-1} &= 0, \end{aligned} \quad (4.78)$$

for $j = 1, 2, 3$, and

$$\begin{aligned} H_h^{n+1} - H_h^n + kP \left[\sum_{j=1}^4 b_j \Phi_h^{n,j-1} \right]_x &= 0, \\ U_h^{n+1} - U_h^n + kA \left[\sum_{j=1}^4 b_j F_h^{n,j-1} \right] &= 0. \end{aligned} \quad (4.79)$$

Finally, we may write the relations (4.75) in the form

$$\begin{aligned} V^{n,j} - H^n + ka_j P \Phi_x^{n,j-1} &= 0, \\ W^{n,j} - U^n + ka_j A F^{n,j-1} &= 0, \end{aligned} \quad (4.80)$$

for $j = 1, 2, 3$.

Lemma 4.5. Suppose that the solution (η, u) of (SCB) is sufficiently smooth and let $\lambda = k/h$. Define, for $0 \leq n \leq M-1$, δ_1^n and δ_2^n by the equations

$$\delta_1^n = H^{n+1} - H^n + kP\left[\sum_{j=1}^4 b_j \Phi^{n,j-1}\right]_x, \quad (4.81)$$

$$\delta_2^n = U^{n+1} - U^n + kA\left[\sum_{j=1}^4 b_j F^{n,j-1}\right]. \quad (4.82)$$

Then, there exists a constant C_λ which is a polynomial of λ with nonnegative coefficients, such that

$$\max_{0 \leq n \leq M-1} \|\delta_1^n\| \leq C_\lambda k(k^4 + h^{3.5} \sqrt{\ln 1/h}), \quad \max_{0 \leq n \leq M-1} \|\delta_2^n\|_1 \leq C_\lambda k(k^4 + h^4).$$

Proof. We need to find formulas for the functions $V^{n,j}$, $W^{n,j}$, for $j = 1, 2, 3$ up to some remainder terms. From the first equation of (4.77) we have

$$V^{n,1} - H^n + a_1 k P \Phi_x^n = 0,$$

whence from (4.73)

$$V^{n,1} = H^n + a_1 k H_t^n - a_1 k \psi^n. \quad (4.83)$$

Also, from the second equation of (4.80) we obtain

$$W^{n,1} - U^n + a_1 k A F^n = 0,$$

and so from (4.77)

$$W^{n,1} = U^n + a_1 k U_t^n - a_1 k \zeta^n. \quad (4.84)$$

Thus

$$V^{n,1} W^{n,1} = H^n U^n + a_1 k (H U)_t^n + a_1^2 k^2 H_t^n U_t^n + v_1^n, \quad (4.85)$$

where, in view of (4.72) and inverse properties

$$\|v_1^n\|_1 \leq C_\lambda h^{3.5} \sqrt{\ln 1/h}.$$

In addition, from (4.84), (4.85) we obtain

$$W^{n,1} + \frac{1}{2} V^{n,1} W^{n,1} = U^n + \frac{1}{2} H^n U^n + a_1 k (U_t^n + \frac{1}{2} (H U)_t^n) + \frac{a_1^2}{2} k^2 H_t^n U_t^n + v_2^n,$$

i.e.

$$\Phi^{n,1} = \Phi^n + a_1 k \Phi_t^n + \frac{a_1^2}{2} k^2 H_t^n U_t^n + v_2^n, \quad (4.86)$$

where

$$\|v_2^n\|_1 \leq C_\lambda h^{3.5} \sqrt{\ln 1/h}.$$

Since now

$$V^{n,2} = H^n - a_2 k P \Phi_x^{n,1},$$

we have from (4.86) that

$$V^{n,2} = H^n - a_2 k P \Phi_x^n - a_1 a_2 k^2 P \Phi_{xt}^n - \frac{a_1^2 a_2}{2} k^3 P (H_t^n U_t^n)_x - a_2 k P (v_2^n)_x,$$

from which, in view of (4.73), (4.72), and inverse properties there follows that

$$V^{n,2} = H^n + a_2 k H_t^n + a_1 a_2 k^2 H_{tt}^n - \frac{a_1^2 a_2}{2} k^3 P (H_t^n U_t^n)_x + \psi_1^n, \quad (4.87)$$

with

$$\|\psi_1^n\| \leq C_\lambda h^4,$$

and

$$\|\psi_1^n\|_1 \leq C_\lambda h^{3.5} \sqrt{\ln 1/h}.$$

In addition, from (4.84)

$$\begin{aligned} W^{n,1} W_x^{n,1} &= (U^n + a_1 k U_t^n - a_1 k \zeta^n)(U_x^n + a_1 k U_{tx}^n - a_1 k \zeta_x^n) \\ &= U^n U_x^n + a_1 k (U U_x)_t^n + a_1^2 k^2 U_t^n U_{tx}^n + w_1^n, \end{aligned}$$

where, in view of (4.72)

$$\|w_1^n\| \leq C k h^4.$$

Similarly, from (4.80) there follows

$$\begin{aligned} V^{n,1}V_x^{n,1} &= (H^n + a_1kH_t^n - a_1k\psi^n)(H_x^n + a_1kH_{tx}^n - a_1k\psi_x^n) \\ &= H^n H_x^n + a_1k(HH_x)_t^n + a_1^2k^2H_t^n H_{tx}^n + w_2^n, \end{aligned}$$

where, in view of (4.72) and inverse properties

$$\|w_2^n\|_{-1} \leq C_\lambda h^4,$$

and

$$\|w_2^n\| \leq C_\lambda h^{3.5} \sqrt{\ln 1/h}.$$

Therefore, from the above relations and (4.83) we have

$$V_x^{n,1} + \frac{3}{2}W^{n,1}W_x^{n,1} + \frac{1}{2}V^{n,1}V_x^{n,1} = F^n + a_1kF_t^n + a_1^2k^2(\frac{3}{2}U_t^n U_{tx}^n + \frac{1}{2}H_t^n H_{tx}^n) + w_3^n,$$

i.e.

$$F^{n,1} = F^n + a_1kF_t^n + a_1^2k^2(\frac{3}{2}U_t^n U_{tx}^n + \frac{1}{2}H_t^n H_{tx}^n) + w_3^n, \quad (4.88)$$

where

$$w_3^n = -k\psi_{1x}^n + \frac{3}{2}w_1^n + \frac{1}{2}w_2^n.$$

Since

$$W^{n,2} = U^n - a_2kAF^{n,1},$$

there follows from (4.88), (4.77), and previous estimates that

$$W^{n,2} = U^n + a_2kU_t^n + a_1a_2k^2U_{tt}^n - a_1^2a_2k^3[\frac{3}{2}A(U_t^n U_{tx}^n) + \frac{1}{2}A(H_t^n H_{tx}^n)] + \zeta_1^n, \quad (4.89)$$

where

$$\|\zeta_1^n\|_1 \leq C_\lambda kh^4.$$

We now have from (4.87), (4.89), the fact that $a_1 = a_2$, and similar estimates to those used above, that

$$\begin{aligned} V^{n,2}W^{n,2} &= H^n U^n + a_2k(HU)_t^n + a_1a_2k^2(H^n U_{tt}^n + H_t^n U_t^n + H_{tt}^n U^n) + a_1a_2^2k^3(H_t^n U_{tt}^n + H_{tt}^n U_t^n) \\ &\quad - a_1^2a_2k^3[\frac{3}{2}H^n A(U_t^n U_{tx}^n) + \frac{1}{2}H^n A(H_t^n H_{tx}^n) - \frac{1}{2}U^n P(H_t^n U_t^n)_x] + v_3^n, \end{aligned}$$

where

$$\|v_3^n\|_1 \leq C_\lambda h^{3.5} \sqrt{\ln 1/h}$$

follows from similar, as above, estimates and from the stability of P in H^1 . Rewriting the previous equation as

$$\begin{aligned} V^{n,2}W^{n,2} &= H^n U^n + a_2k(HU)_t^n + a_1a_2k^2(HU)_{tt}^n - a_1a_2k^2H_t^n U_t^n + a_1a_2^2k^3(H_t U_t)_t^n \\ &\quad - a_1^2a_2k^3[\frac{3}{2}H^n A(U_t^n U_{tx}^n) + \frac{1}{2}H^n A(H_t^n H_{tx}^n) + \frac{1}{2}U^n P(H_t^n U_t^n)_x] + v_3^n, \end{aligned}$$

we have, by the definition of $\Phi^{n,2}$ and (4.89)

$$\begin{aligned} \Phi^{n,2} &= \Phi^n + a_2k\Phi_t^n + a_1a_2k^2\Phi_{tt}^n - \frac{a_1a_2}{2}k^2H_t^n U_t^n + \frac{a_1a_2^2}{2}k^3(H_t U_t)_t^n \\ &\quad - a_1^2a_2k^3[\frac{3}{2}A(U_t^n U_{tx}^n) + \frac{1}{2}A(H_t^n H_{tx}^n) + \frac{3}{4}H^n A(U_t^n U_{tx}^n) + \frac{1}{4}H^n A(H_t^n H_{tx}^n) \\ &\quad + \frac{1}{4}U^n P(H_t^n U_t^n)_x] + \zeta_1^n + \frac{1}{2}v_3^n. \end{aligned} \quad (4.90)$$

Since now

$$V^{n,3} = H^n - a_3kP\Phi_x^{n,2} = H^n - kP\Phi_x^{n,2},$$

we have from (4.90) and (4.76) that

$$V^{n,3} = H^n + kH_t^n + a_2k^2H_{tt}^n + a_1a_2k^3H_{ttt}^n + \frac{a_1a_2}{2}k^3P(H_t^n U_t^n)_x + \psi_2^n, \quad (4.91)$$

where

$$\|\psi_2^n\| \leq C_\lambda h^4,$$

and (using the fact that $\|Af\|_2 \leq C\|f\|$ for $f \in L^2$),

$$\|\psi_2^n\|_1 \leq C_\lambda h^{3.5} \sqrt{\ln 1/h}.$$

In addition, from (4.89)

$$W^{n,2}W_x^{n,2} = \left[U^n + a_2kU_t^n + a_1a_2k^2U_{tt}^n - a_1^2a_2k^3 \left[\frac{3}{2}A(U_t^nU_{tx}^n) + \frac{1}{2}A(H_t^nH_{tx}^n) \right] + \zeta_1^n \right] \cdot \left[U_x^n + a_2kU_{tx}^n + a_1a_2k^2U_{ttx}^n - a_1^2a_2k^3 \left[\frac{3}{2}A(U_t^nU_{tx}^n) + \frac{1}{2}A(H_t^nH_{tx}^n) \right]_x + \zeta_{1x}^n \right],$$

and so

$$W^{n,2}W_x^{n,2} = U^nU_x^n + a_2k(UU_x)_t^n + a_1a_2k^2(UU_x)_{tt}^n - a_1a_2k^2U_t^nU_{tx}^n + a_1a_2^2k^3(U_t^nU_{tt}^n)_x - a_1^2a_2k^3 \left[\frac{3}{2}U^nA(U_t^nU_{tx}^n) + \frac{1}{2}U^nA(H_t^nH_{tx}^n) \right]_x + w_4^n, \quad (4.92)$$

where $\|w_4^n\| \leq C_\lambda h^4$. In addition, from (4.87)

$$V^{n,2}V_x^{n,2} = \left[H^n + a_2kH_t^n + a_1a_2k^2H_{tt}^n - \frac{a_1^2a_2}{2}k^3P(H_t^nU_t^n)_x + \psi_1^n \right] \cdot \left[H_x^n + a_2kH_{tx}^n + a_1a_2k^2H_{ttx}^n - \frac{a_1^2a_2}{2}k^3(P(H_t^nU_t^n)_x)_x + \psi_{1x}^n \right].$$

Hence,

$$V^{n,2}V_x^{n,2} = H^nH_x^n + a_2k(HH_x)_t^n + a_1a_2k^2(HH_x)_{tt}^n - a_1a_2k^2H_t^nH_{tx}^n + a_1a_2^2k^3(H_t^nH_{tt}^n)_x - \frac{a_1^2a_2}{2}k^3[H^nP(H_t^nU_t^n)_x]_x + v_4^n, \quad (4.93)$$

where

$$\|v_4^n\|_{-1} \leq C_\lambda h^4,$$

and

$$\|v_4^n\| \leq C_\lambda h^{3.5} \sqrt{\ln 1/h}.$$

From the definition of $F^{n,2}$, (4.87), (4.92) and (4.93) we have

$$F^{n,2} = F^n + a_2kF_t^n + a_1a_2k^2F_{tt}^n - a_1a_2k^2 \left(\frac{3}{2}U_t^nU_{tx}^n + \frac{1}{2}H_t^nH_{tx}^n \right) + a_1a_2^2k^3 \left[\frac{3}{2}U_t^nU_{tt}^n + \frac{1}{2}H_t^nH_{tt}^n - \frac{1}{2}P(H_t^nU_t^n)_x - \frac{9}{4}U^nA(U_t^nU_{tx}^n) - \frac{3}{4}U^nA(H_t^nH_{tx}^n) - \frac{1}{4}H^nP(H_t^nU_t^n)_x \right]_x + w_5^n, \quad (4.94)$$

where

$$w_5^n = \psi_{1x}^n + \frac{3}{2}w_4^n + \frac{1}{2}v_4^n.$$

Hence, since

$$W^{n,3} = U^n - kAF^{n,2},$$

we obtain

$$W^{n,3} = U^n + kU_t^n + a_2k^2U_{tt}^n + a_1a_2k^3U_{ttt}^n + a_1a_2k^3 \left[\frac{3}{2}A(U_t^nU_{tx}^n) + \frac{1}{2}A(H_t^nH_{tx}^n) \right] + \zeta_2^n, \quad (4.95)$$

where

$$\|\zeta_2^n\|_1 \leq C_\lambda kh^4 + Ck^4.$$

From (4.91), (4.95) we get

$$V^{n,3}W^{n,3} = H^nU^n + k(HU)_t^n + a_2k^2(HU)_{tt}^n + a_1a_2k^3(HU)_{ttt}^n - a_1a_2k^3(H_tU_t)_t^n + a_1a_2k^3 \left[\frac{3}{2}H^nA(U_t^nU_{tx}^n) + \frac{1}{2}H^nA(H_t^nH_{tx}^n) + \frac{1}{2}U^nP(H_t^nU_t^n)_x \right] + v_5^n,$$

where

$$\|v_5^n\|_1 \leq C_\lambda h^{3.5} \sqrt{\ln 1/h}.$$

Hence, from the definition of $\Phi^{n,3}$, (4.95) and the last relation, we see that

$$\Phi^{n,3} = \Phi^n + k\Phi_t^n + a_2k^2\Phi_{tt}^n + a_1a_2k^3\Phi_{ttt}^n - \frac{a_1a_2}{2}k^3(H_tU_t)_t^n + a_1a_2k^3 \left[\frac{3}{2}A(U_t^nU_{tx}^n) + \frac{1}{2}A(H_t^nH_{tx}^n) + \frac{3}{4}H^nA(U_t^nU_{tx}^n) + \frac{1}{4}H^nA(H_t^nH_{tx}^n) + \frac{1}{4}U^nP(H_t^nU_t^n)_x \right] + \zeta_2^n + \frac{1}{2}v_5^n. \quad (4.96)$$

We are now ready to estimate δ_1^n . By (4.90) and (4.96) we have

$$b_3\Phi^{n,2} + b_4\Phi^{n,3} = (b_3 + b_4)\Phi^n + (b_3a_2 + b_4)k\Phi_t^n + (a_1a_2b_3 + a_2b_4)k^2\Phi_{tt}^n + a_1a_2b_4k^3\Phi_{ttt}^n - \frac{a_1a_2b_3}{2}k^2H_t^nU_t^n + b_3(\zeta_1^n + \frac{1}{2}v_3^n) + b_4(\zeta_2^n + \frac{1}{2}v_5^n).$$

In addition, in view of (4.86)

$$b_1\Phi^n + b_2\Phi^{n,1} = (b_1 + b_2)\Phi^n + a_1b_2k\Phi_t^n + \frac{a_1^2b_2}{2}k^2H_t^nU_t^n + b_2v_2^n.$$

Hence,

$$\sum_{j=1}^4 b_j\Phi^{n,j-1} = \Phi^n + \frac{k}{2}\Phi_t^n + \frac{k^2}{6}\Phi_{tt}^n + \frac{k^3}{24}\Phi_{ttt}^n + v_6^n,$$

where

$$v_6^n = b_2v_2^n + b_3(\zeta_1^n + \frac{1}{2}v_3^n) + b_4(\zeta_2^n + \frac{1}{2}v_5^n).$$

Therefore, by (4.81) and (4.76) we obtain

$$\delta_1^n = H^{n+1} - H^n - kH_t^n - \frac{k^2}{2}H_{tt}^n - \frac{k^3}{6}H_{ttt}^n - \frac{k^4}{24}H_{tttt}^n + kP(v_6^n)_x + v_7^n,$$

where, by (4.72)

$$\|v_7^n\| \leq C_\lambda kh^{3.5}\sqrt{\ln 1/h}.$$

Since, in addition

$$\|v_6^n\|_1 \leq C_\lambda h^{3.5}\sqrt{\ln 1/h},$$

we conclude that

$$\|\delta_1^n\| \leq C_\lambda k(k^4 + h^{3.5}\sqrt{\ln 1/h}).$$

To estimate δ_2^n we must also compute $F^{n,3}$. To this end, using (4.95) we have after some algebra

$$\begin{aligned} W^{n,3}W_x^{n,3} &= U^nU_x^n + k(UU_x)_t^n + a_2k^2(UU_x)_{tt}^n + a_1a_2k^3(UU_x)_{ttt}^n - a_1a_2k^3(U_t^nU_{tt}^n)_x \\ &\quad + a_1a_2k^3[\frac{3}{2}U^nA(U_t^nU_{tx}^n) + \frac{1}{2}U^nA(H_t^nH_{tx}^n)]_x + w_6^n, \end{aligned} \quad (4.97)$$

where

$$\|w_6^n\| \leq C_\lambda h^4.$$

Similarly, from (4.91), we have

$$\begin{aligned} V^{n,3}V_x^{n,3} &= H^nH_x^n + k(HH_x)_t^n + a_2k^2(HH_x)_{tt}^n + a_1a_2k^3(HH_x)_{ttt}^n \\ &\quad - a_1a_2k^3(H_t^nH_{tt}^n)_x + \frac{a_1a_2}{2}k^3[H^nP(H_t^nU_t^n)_x]_x + v_7^n, \end{aligned}$$

where

$$\|v_7^n\|_{-1} \leq C_\lambda h^4, \quad \|v_7^n\| \leq C_\lambda h^{3.5}\sqrt{\ln 1/h}.$$

Hence, from the definition of $F^{n,3}$, (4.91), (4.97) and the last relation, we obtain

$$\begin{aligned} F^{n,3} &= F^n + kF_t^n + a_2k^2F_{tt}^n + a_1a_2k^3F_{ttt}^n - a_1a_2k^3[\frac{3}{2}U_t^nU_{tt}^n + \frac{1}{2}H_t^nH_{tt}^n]_x \\ &\quad + a_1a_2k^3[\frac{1}{2}P(H_t^nU_t^n)_x + \frac{9}{4}U^nA(U_t^nU_{tx}^n) + \frac{3}{4}U^nA(H_t^nH_{tx}^n) \\ &\quad + \frac{1}{4}H^nP(H_t^nU_t^n)_x]_x + w_7^n, \end{aligned} \quad (4.98)$$

where

$$w_7^n = \psi_{2x}^n + \frac{3}{2}w_6^n + \frac{1}{2}v_7^n.$$

We may now estimate δ_2^n . By (4.94) and (4.98) we obtain

$$\begin{aligned} b_3F^{n,2} + b_4F^{n,3} &= (b_3 + b_4)F^n + (a_2b_3 + b_4)kF_t^n + (a_1a_2b_3 + a_2b_4)k^2F_{tt}^n + a_1a_2b_4k^3F_{ttt}^n \\ &\quad - a_1a_2b_3k^2(\frac{3}{2}U_t^nU_{tx}^n + \frac{1}{2}H_t^nH_{tx}^n) + b_3w_5^n + b_4w_7^n. \end{aligned}$$

Also, in view of (4.88)

$$b_1F^n + b_2F^{n,1} = (b_1 + b_2)F^n + a_1b_2kF_t^n + a_1^2b_2k^2(\frac{3}{2}U_t^nU_{tx}^n + \frac{1}{2}H_t^nH_{tx}^n) + b_2w_3^n.$$

Therefore

$$\sum_{j=1}^4 b_jF^{n,j-1} = F^n + \frac{k}{2}F_t^n + \frac{k^2}{6}F_{tt}^n + \frac{k^3}{24}F_{ttt}^n + w_8^n,$$

where

$$w_8^n = b_2w_3^n + b_3w_5^n + b_4w_7^n.$$

Hence, from (4.82) and (4.77) we obtain

$$\delta_2^n = U^{n+1} - U^n - kU_t^n - \frac{k^2}{2}U_{tt}^n - \frac{k^3}{6}U_{ttt}^n - \frac{k^4}{24}U_{tttt}^n + kAw_8^n + w_9^n,$$

where, by (4.72)

$$\|w_9^n\|_1 \leq Ckh^4.$$

Since we also have, by previous estimates,

$$\|w_8^n\|_{-1} \leq C_\lambda h^4,$$

we finally conclude that

$$\|\delta_2^n\|_1 \leq C_\lambda k(k^4 + h^4),$$

and the proof of the Lemma 4.5 is completed. \square

We show stability and convergence of the fully discrete scheme in the course of the proof of the following result.

Proposition 4.3. *Suppose that the solution (η, u) of (SCB) is sufficiently smooth on $[0, T]$. Let $\lambda = k/h$ and (H_h^n, U_h^n) be the solution of (4.73)-(4.74). Then, there exists a positive constant λ_0 and a constant C independent of k and h , such that for $\lambda \leq \lambda_0$,*

$$\max_{0 \leq n \leq M} \|\eta(t^n) - H_h^n\| \leq C(k^4 + h^{3.5} \sqrt{\ln 1/h}), \quad \max_{0 \leq n \leq M} \|u(t^n) - U_h^n\|_1 \leq C(k^4 + h^3).$$

Proof. It suffices to show that

$$\max_{0 \leq n \leq M} (\|H^n - H_h^n\| + \|U^n - U_h^n\|_1) \leq C(k^4 + h^{3.5} \sqrt{\ln 1/h}).$$

Let

$$\varepsilon^n = H^n - H_h^n, \quad e^n = U^n - U_h^n, \quad \varepsilon^{n,j} = V^{n,j} - H_h^{n,j}, \quad e^{n,j} = W^{n,j} - U_h^{n,j}, \quad j = 1, 2, 3.$$

Then, by (4.75) and (4.78), we have for $j = 1, 2, 3$

$$\varepsilon^{n,j} = \varepsilon^n - ka_j P(\Phi^{n,j-1} - \Phi_h^{n,j-1})_x, \quad (4.99)$$

$$e^{n,j} = e^n - ka_j A(F^{n,j-1} - F_h^{n,j-1}), \quad (4.100)$$

and by (4.79), (4.81), and (4.82)

$$\varepsilon^{n+1} = \varepsilon^n - k \sum_{j=1}^4 b_j P(\Phi^{n,j-1} - \Phi_h^{n,j-1})_x + \delta_1^n, \quad (4.101)$$

$$e^{n+1} = e^n - k \sum_{j=1}^4 b_j A(F^{n,j-1} - F_h^{n,j-1}) + \delta_2^n. \quad (4.102)$$

We embark upon the proof by deriving suitable estimates for $\varepsilon^{n,1}$ and $e^{n,1}$. To this end, putting $\Phi_h^n := \Phi_h^{n,0}$, we have

$$\Phi^n - \Phi_h^n = e^n + \frac{1}{2}(H^n U^n - H_h^n U_h^n),$$

and since

$$\begin{aligned} H^n U^n - H_h^n U_h^n &= H^n(U^n - U_h^n) - (U^n - U_h^n)(H^n - H_h^n) + U^n(H^n - H_h^n) \\ &= H^n e^n - \varepsilon^n e^n + U^n \varepsilon^n, \end{aligned}$$

we will have

$$\Phi^n - \Phi_h^n = e^n + \frac{1}{2}(H^n e^n - \varepsilon^n e^n) + \frac{1}{2}U^n \varepsilon^n,$$

i.e.

$$\Phi^n - \Phi_h^n = \tilde{\gamma}_1^n + \frac{1}{2}U^n \varepsilon^n,$$

where

$$\tilde{\gamma}_1^n = e^n + \frac{1}{2}(H^n e^n - \varepsilon^n e^n).$$

Therefore,

$$P(\Phi^n - \Phi_h^n)_x = \gamma_1^n + \frac{1}{2}\rho_1^n, \quad (4.103)$$

where

$$\gamma_1^n = P\tilde{\gamma}_{1x}^n, \quad \rho_1^n = P(U^n \varepsilon^n)_x. \quad (4.104)$$

Hence, from (4.99)

$$\varepsilon^{n,1} = \varepsilon^n - a_1 k \gamma_1^n - \frac{a_1}{2} k \rho_1^n. \quad (4.105)$$

Let $0 \leq n^* \leq M-1$ be the maximal integer for which

$$\|\varepsilon^n\|_1 + \|e^n\|_1 \leq 1, \quad 0 \leq n \leq n^*.$$

Then, for $n \leq n^*$,

$$\|\tilde{\gamma}_1^n\|_1 \leq C(\|\varepsilon^n\| + \|e^n\|_1),$$

and, in view of (4.104),

$$\|\gamma_1^n\| \leq C(\|\varepsilon^n\| + \|e^n\|_1). \quad (4.106)$$

Therefore, by (4.99), for $0 \leq n \leq n^*$,

$$\|\varepsilon^{n,1}\| \leq C_\lambda(\|\varepsilon^n\| + \|e^n\|_1), \quad (4.107)$$

and also

$$\|\varepsilon^{n,1}\|_1 \leq 1 + C_\lambda \|\gamma_1^n\| + C_\lambda \|\rho_1^n\| \leq C_\lambda, \quad (4.108)$$

where, as before, C_λ denotes a constant that is a polynomial of λ with positive coefficients.

In order to estimate $e^{n,1}$ we note, setting $F_h^n = F_h^{n,0}$, that

$$F^n - F_h^n = \varepsilon_x^n + \frac{3}{2}(U^n U_x^n - U_h^n U_{hx}^n) + \frac{1}{2}(H^n H_x^n - H_h^n H_{hx}^n).$$

Since

$$\begin{aligned} U^n U_x^n - U_h^n U_{hx}^n &= U^n(U_x^n - U_{hx}^n) - (U_x^n - U_{hx}^n)(U^n - U_h^n) + U_x^n(U^n - U_h^n) \\ &= (U^n e^n)_x - e^n e_x^n, \end{aligned}$$

and

$$H^n H_x^n - H_h^n H_{hx}^n = (H^n \varepsilon^n)_x - \varepsilon^n \varepsilon_x^n,$$

we have

$$F^n - F_h^n = \varepsilon_x^n + \frac{3}{2}(U^n e^n)_x - \frac{3}{2}e^n e_x^n + \frac{1}{2}(H^n \varepsilon^n)_x - \frac{1}{2}\varepsilon^n \varepsilon_x^n,$$

and therefore, for $n \leq n^*$

$$\|F^n - F_h^n\|_{-1} \leq C(\|\varepsilon^n\| + \|e^n\|). \quad (4.109)$$

Recalling the expression for $e^{n,1}$ in (4.101) we have for $n \leq n^*$

$$\|e^{n,1}\|_1 \leq C(\|\varepsilon^n\| + \|e^n\|_1). \quad (4.110)$$

We now turn to deriving estimates for $\varepsilon^{n,2}$ and $e^{n,2}$. With this aim in mind, we note for later reference that it follows from the definition of $V^{n,j}$, $W^{n,j}$, $F^{n,j}$ that for $0 \leq n \leq M-1$, $j = 0, 1, 2, 3$

$$\|V^{n,j}\|_1 + \|W^{n,j}\|_1 \leq C_\lambda, \quad \|F^{n,j}\|_{-1} \leq C_\lambda.$$

Since now

$$\Phi^{n,1} - \Phi_h^{n,1} = e^{n,1} + \frac{1}{2}(V^{n,1}W^{n,1} - H_h^{n,1}U_h^{n,1}),$$

and

$$\begin{aligned} V^{n,1}W^{n,1} - H_h^{n,1}U_h^{n,1} &= V^{n,1}(W^{n,1} - U_h^{n,1}) - (W^{n,1} - U_h^{n,1})(V^{n,1} - H_h^{n,1}) + W^{n,1}(V^{n,1} - H_h^{n,1}) \\ &= V^{n,1}e^{n,1} - e^{n,1}\varepsilon^{n,1} + W^{n,1}\varepsilon^{n,1}, \end{aligned}$$

using (4.80) we have by (4.105)

$$\begin{aligned} W^{n,1}\varepsilon^{n,1} &= (U^n - a_1 k A F^n)(\varepsilon^n - a_1 k \gamma_1^n - \frac{a_1 k}{2} \rho_1^n) \\ &= U^n \varepsilon^n - \frac{a_1 k}{2} U^n \rho_1^n - a_1 k U^n \gamma_1^n - a_1 k (A F^n) \varepsilon^{n,1}, \end{aligned}$$

and so

$$\Phi^{n,1} - \Phi_h^{n,1} = \frac{1}{2}U^n \varepsilon^n - \frac{a_1 k}{4}U^n \rho_1^n + \tilde{\gamma}_2^n,$$

where

$$\tilde{\gamma}_2^n = e^{n,1} + \frac{1}{2}V^{n,1}e^{n,1} - \frac{1}{2}e^{n,1}\varepsilon^{n,1} - \frac{a_1 k}{2}U^n \gamma_1^n - \frac{a_1 k}{2}(A F^n)\varepsilon^{n,1}. \quad (4.111)$$

Recalling the definition of ρ_1^n in (4.104) we obtain

$$P(\Phi^{n,1} - \Phi_h^{n,1})_x = \frac{1}{2}\rho_1^n - \frac{a_1 k}{4}\rho_2^n + \gamma_2^n, \quad (4.112)$$

where

$$\gamma_2^n = P(\tilde{\gamma}_2^n)_x, \quad \rho_2^n = P(U^n \rho_1^n)_x. \quad (4.113)$$

By (4.111) and (4.108) we deduce that

$$\|\tilde{\gamma}_2^n\|_1 \leq C_\lambda \|e^{n,1}\|_1 + Ck \|\gamma_1^n\|_1 + C_\lambda k \|\varepsilon^{n,1}\|_1.$$

Hence, by (4.110) and (4.106), (4.107) we have

$$\|\tilde{\gamma}_2^n\|_1 \leq C_\lambda (\|\varepsilon^n\| + \|e^n\|_1),$$

and therefore, by (4.113) we see that

$$\|\gamma_2^n\| \leq C_\lambda (\|\varepsilon^n\| + \|e^n\|_1), \quad (4.114)$$

holds for $n \leq n^*$. Now, by (4.80) and (4.78)

$$\varepsilon^{n,2} = V^{n,2} - H_h^{n,2} = \varepsilon^n - a_2 k P(\Phi^{n,1} - \Phi_h^{n,1})_x,$$

and from (4.112)

$$\varepsilon^{n,2} = \varepsilon^n - \frac{a_2 k}{2}\rho_1^n + \frac{a_1 a_2 k^2}{4}\rho_2^n - a_2 k \gamma_2^n. \quad (4.115)$$

Thus,

$$\|\varepsilon^{n,2}\| \leq \|\varepsilon^n\| + Ck \|\rho_1^n\| + Ck^2 \|\rho_2^n\| + Ck \|\gamma_2^n\|,$$

and using (4.113), (4.104) and (4.114) we conclude that for $n \leq n^*$

$$\|\varepsilon^{n,2}\| \leq C_\lambda (\|\varepsilon^n\| + \|e^n\|_1), \quad (4.116)$$

and

$$\|\varepsilon^{n,2}\|_1 \leq C_\lambda. \quad (4.117)$$

We next consider $e^{n,2}$. Since

$$F^{n,1} - F_h^{n,1} = \varepsilon_x^{n,1} + \frac{3}{2}(W^{n,1}W_x^{n,1} - U_h^{n,1}U_{hx}^{n,1}) + \frac{1}{2}(V^{n,1}V_x^{n,1} - H_h^{n,1}H_{hx}^{n,1}),$$

$$\begin{aligned} W^{n,1}W_x^{n,1} - U_h^{n,1}U_{hx}^{n,1} &= W^{n,1}(W_x^{n,1} - U_{hx}^{n,1}) - (W_x^{n,1} - U_{hx}^{n,1})(W^{n,1} - U_h^{n,1}) + W_x^{n,1}(W^{n,1} - U_h^{n,1}) \\ &= (W^{n,1}e^{n,1})_x - e_x^{n,1}e^{n,1}, \end{aligned}$$

and

$$V^{n,1}V_x^{n,1} - H_h^{n,1}H_{hx}^{n,1} = (V^{n,1}e^{n,1})_x - \varepsilon_x^{n,1}e^{n,1},$$

we obtain by (4.108), for $n \leq n^*$

$$\|F^{n,1} - F_h^{n,1}\|_{-1} \leq C_\lambda (\|\varepsilon^{n,1}\| + \|e^{n,1}\|_1),$$

giving, in view of (4.107) and (4.110) for $n \leq n^*$

$$\|F^{n,1} - F_h^{n,1}\|_{-1} \leq C_\lambda (\|\varepsilon^n\| + \|e^n\|_1). \quad (4.118)$$

Therefore, by (4.100) we conclude that for $n \leq n^*$

$$\|e^{n,2}\|_1 \leq C_\lambda (\|\varepsilon^n\| + \|e^n\|_1). \quad (4.119)$$

To obtain analogous bounds for $\varepsilon^{n,3}$, $e^{n,3}$ note that

$$\Phi^{n,2} - \Phi_h^{n,2} = e^{n,2} + \frac{1}{2}(V^{n,2}W^{n,2} - H_h^{n,2}U_h^{n,2}),$$

and

$$\begin{aligned} V^{n,2}W^{n,2} - H_h^{n,2}U_h^{n,2} &= V^{n,2}(W^{n,2} - U_h^{n,2}) - (W^{n,2} - U_h^{n,2})(V^{n,2} - H_h^{n,2}) + W^{n,2}(V^{n,2} - H_h^{n,2}) \\ &= V^{n,2}e^{n,2} - e^{n,2}\varepsilon^{n,2} + W^{n,2}\varepsilon^{n,2}. \end{aligned}$$

By (4.80) and (4.115) we see that

$$\begin{aligned} W^{n,2}\varepsilon^{n,2} &= (U^n - a_2 k A F^{n,1})(\varepsilon^n - \frac{a_2 k}{2}\rho_1^n + \frac{a_1 a_2 k^2}{4}\rho_2^n - a_2 k \gamma_2^n) \\ &= U^n \varepsilon^n - \frac{a_2 k}{2}U^n \rho_1^n + \frac{a_1 a_2 k^2}{4}U^n \rho_2^n - a_2 k U^n \gamma_2^n - a_2 k (A F^{n,1})\varepsilon^{n,2}. \end{aligned}$$

Therefore

$$\Phi^{n,2} - \Phi_h^{n,2} = \frac{1}{2}U^n \varepsilon^n - \frac{a_2 k}{4}U^n \rho_1^n + \frac{a_1 a_2 k^2}{8}U^n \rho_2^n + \tilde{\gamma}_3^n,$$

where

$$\tilde{\gamma}_3^n = e^{n,2} + \frac{1}{2}V^{n,2}e^{n,2} - \frac{1}{2}e^{n,2}\varepsilon^{n,2} - \frac{a_2 k}{2}U^n \gamma_2^n - \frac{a_2 k}{2}(AF^{n,1})\varepsilon^{n,2}. \quad (4.120)$$

So, by (4.104) and (4.113) we have

$$P(\Phi^{n,2} - \Phi_h^{n,2})_x = \frac{1}{2}\rho_1^n - \frac{a_2 k}{4}\rho_2^n + \frac{a_1 a_2 k^2}{8}\rho_3^n + \gamma_3^n, \quad (4.121)$$

where

$$\gamma_3^n = P(\tilde{\gamma}_3^n)_x, \quad \rho_3^n = P(U^n \rho_2^n)_x. \quad (4.122)$$

Now, by (4.120) and (4.117) we have for $n \leq n^*$

$$\|\tilde{\gamma}_3^n\|_1 \leq C_\lambda \|e^{n,2}\|_1 + Ck \|\gamma_2^n\|_1 + C_\lambda k \|\varepsilon^{n,2}\|_1.$$

Hence, by (4.119), (4.114) and (4.116),

$$\|\tilde{\gamma}_3^n\|_1 \leq C_\lambda (\|\varepsilon^n\| + \|e^n\|_1),$$

and thus

$$\|\gamma_3^n\| \leq C_\lambda (\|\varepsilon^n\| + \|e^n\|_1), \quad (4.123)$$

for $n \leq n^*$. Now

$$\varepsilon^{n,3} = V^{n,3} - H_h^{n,3} = \varepsilon^n - kP(\Phi^{n,2} - \Phi_h^{n,2})_x,$$

and from (4.121)

$$\varepsilon^{n,3} = \varepsilon^n - \frac{k}{2}\rho_1^n + \frac{a_2 k^2}{4}\rho_2^n - \frac{a_1 a_2 k^3}{8}\rho_3^n - k\gamma_3^n. \quad (4.124)$$

Therefore,

$$\|\varepsilon^{n,3}\| \leq \|\varepsilon^n\| + Ck \|\rho_1^n\| + Ck^2 \|\rho_2^n\| + Ck^3 \|\rho_3^n\| + Ck \|\gamma_3^n\|,$$

and by (4.122), (4.113), (4.104) and (4.123) we deduce that for $n \leq n^*$

$$\|\varepsilon^{n,3}\| \leq C_\lambda (\|\varepsilon^n\| + \|e^n\|_1), \quad (4.125)$$

and also

$$\|\varepsilon^{n,3}\|_1 \leq C_\lambda. \quad (4.126)$$

We consider now $e^{n,3}$. Since

$$F^{n,2} - F_h^{n,2} = \varepsilon_x^{n,2} + \frac{3}{2}(W^{n,2}W_x^{n,2} - U_h^{n,2}U_{hx}^{n,2}) + \frac{1}{2}(V^{n,2}V_x^{n,2} - H_h^{n,2}H_{hx}^{n,2}),$$

$$\begin{aligned} W^{n,2}W_x^{n,2} - U_h^{n,2}U_{hx}^{n,2} &= W^{n,2}(W_x^{n,2} - U_{hx}^{n,2}) - (W_x^{n,2} - U_{hx}^{n,2})(W^{n,2} - U_h^{n,2}) + W_x^{n,2}(W^{n,2} - U_h^{n,2}) \\ &= (W^{n,2}e^{n,2})_x - e_x^{n,2}e^{n,2}, \end{aligned}$$

and

$$V^{n,2}V_x^{n,2} - H_h^{n,2}H_{hx}^{n,2} = (V^{n,2}\varepsilon^{n,2})_x - \varepsilon_x^{n,2}\varepsilon^{n,2},$$

we have by (4.117) for $n \leq n^*$

$$\|F^{n,2} - F_h^{n,2}\|_{-1} \leq C_\lambda (\|\varepsilon^{n,2}\| + \|e^{n,2}\|_1),$$

and from (4.116) and (4.119), for $n \leq n^*$

$$\|F^{n,2} - F_h^{n,2}\|_{-1} \leq C_\lambda (\|\varepsilon^n\| + \|e^n\|_1). \quad (4.127)$$

Therefore, we conclude by (4.100), for $n \leq n^*$

$$\|e^{n,3}\|_1 \leq C_\lambda (\|\varepsilon^n\| + \|e^n\|_1). \quad (4.128)$$

We have now reached the final stage of the estimation of the errors of the fully discrete scheme. First, we estimate e^{n+1} : Since

$$F^{n,3} - F_h^{n,3} = \varepsilon_x^{n,3} + \frac{3}{2}(W^{n,3}W_x^{n,3} - U_h^{n,3}U_{hx}^{n,3}) + \frac{1}{2}(V^{n,3}V_x^{n,3} - H_h^{n,3}H_{hx}^{n,3}),$$

$$W^{n,3}W_x^{n,3} - U_h^{n,3}U_{hx}^{n,3} = (W^{n,3}e^{n,3})_x - e_x^{n,3}e^{n,3},$$

and

$$V^{n,3}V_x^{n,3} - H_h^{n,3}H_{hx}^{n,3} = (V^{n,3}\varepsilon^{n,3})_x - \varepsilon_x^{n,3}\varepsilon^{n,3},$$

we have from (4.126) for $n \leq n^*$,

$$\|F^{n,3} - F_h^{n,3}\|_{-1} \leq C_\lambda(\|\varepsilon^{n,3}\| + \|e^{n,3}\|_1),$$

and therefore, from (4.125), (4.128)

$$\|F^{n,3} - F_h^{n,3}\|_{-1} \leq C_\lambda(\|\varepsilon^n\| + \|e^n\|_1).$$

Thus, by (4.102), (4.109), (4.118), (4.127), and above inequality, we obtain for $n \leq n^*$

$$\|e^{n+1}\|_1 \leq \|e^n\|_1 + C_\lambda k(\|\varepsilon^n\| + \|e^n\|_1) + \|\delta_2^n\|_1, \quad (4.129)$$

which is the required recursive inequality for $\|e^{n+1}\|_1$. To obtain an analogous relation for $\|\varepsilon^{n+1}\|$ requires more work. Observe that

$$\Phi^{n,3} - \Phi_h^{n,3} = e^{n,3} + \frac{1}{2}(V^{n,3}W^{n,3} - H_h^{n,3}U_h^{n,3}),$$

and

$$\begin{aligned} V^{n,3}W^{n,3} - H_h^{n,3}U_h^{n,3} &= V^{n,3}(W^{n,3} - U_h^{n,3}) - (W^{n,3} - U_h^{n,3})(V^{n,3} - H_h^{n,3}) + W^{n,3}(V^{n,3} - H_h^{n,3}) \\ &= V^{n,3}e^{n,3} - e^{n,3}\varepsilon^{n,3} + W^{n,3}\varepsilon^{n,3}. \end{aligned}$$

By (4.80) and (4.124) we obtain

$$\begin{aligned} W^{n,3}\varepsilon^{n,3} &= (U^n - kAF^{n,2})(\varepsilon^n - \frac{k}{2}\rho_1^n + \frac{a_2k^2}{4}\rho_2^n - \frac{a_1a_2k^3}{8}\rho_3^n - k\gamma_3^n) \\ &= U^n\varepsilon^n - \frac{k}{2}U^n\rho_1^n + \frac{a_2k^2}{4}U^n\rho_2^n - \frac{a_1a_2k^3}{8}U^n\rho_3^n - kU^n\gamma_3^n - k(AF^{n,2})\varepsilon^{n,3}. \end{aligned}$$

Thus

$$\Phi^{n,3} - \Phi_h^{n,3} = \frac{1}{2}U^n\varepsilon^n - \frac{k}{4}U^n\rho_1^n + \frac{a_2k^2}{8}U^n\rho_2^n - \frac{a_1a_2k^3}{16}U^n\rho_3^n + \tilde{\gamma}_4^n,$$

where

$$\tilde{\gamma}_4^n = e^{n,3} + \frac{1}{2}V^{n,3}e^{n,3} - \frac{1}{2}e^{n,3}\varepsilon^{n,3} - \frac{k}{2}U^n\gamma_3^n - \frac{k}{2}(AF^{n,2})\varepsilon^{n,3}. \quad (4.130)$$

Hence, by (4.104), (4.113), and (4.132) we obtain

$$P(\Phi^{n,3} - \Phi_h^{n,3})_x = \frac{1}{2}\rho_1^n - \frac{k}{4}\rho_2^n + \frac{a_2k^2}{8}\rho_3^n - \frac{a_1a_2k^3}{16}\rho_4^n + \gamma_4^n, \quad (4.131)$$

where

$$\gamma_4^n = P(\tilde{\gamma}_4^n)_x, \quad \rho_4^n = P(U^n\rho_3^n)_x. \quad (4.132)$$

By (4.130) and (4.126) we obtain for $n \leq n^*$

$$\|\tilde{\gamma}_4^n\|_1 \leq C_\lambda\|e^{n,3}\|_1 + Ck\|\gamma_3^n\|_1 + C_\lambda k\|\varepsilon^{n,3}\|_1,$$

from which, using (4.128), (4.123) and (4.125) we see that for $n \leq n^*$

$$\|\tilde{\gamma}_4^n\|_1 \leq C_\lambda(\|\varepsilon^n\| + \|e^n\|_1),$$

and

$$\|\gamma_4^n\| \leq C_\lambda(\|\varepsilon^n\| + \|e^n\|_1). \quad (4.133)$$

From (4.103) and (4.112) we obtain

$$\begin{aligned} b_1P(\Phi^n - \Phi_h^n)_x + b_2P(\Phi^{n,1} - \Phi_h^{n,1})_x &= \frac{1}{2}(b_1 + b_2)\rho_1^n - \frac{a_1b_2k}{4}\rho_2^n + b_1\gamma_1 + b_2\gamma_2 \\ &= \frac{1}{4}\rho_1^n - \frac{k}{24}\rho_2^n + \frac{1}{6}\gamma_1^n + \frac{1}{3}\gamma_2^n. \end{aligned}$$

From (4.121) and (4.131) we see that

$$\begin{aligned} b_3P(\Phi^{n,2} - \Phi_h^{n,2})_x + b_4P(\Phi^{n,3} - \Phi_h^{n,3})_x &= \frac{1}{2}(b_3 + b_4)\rho_1^n - \frac{(b_3a_2 + b_4)k}{4}\rho_2^n + \frac{(b_3a_1a_2 + b_4a_2)k^2}{8}\rho_3^n \\ &\quad - \frac{b_4a_1a_2k^3}{16}\rho_4^n + b_3\gamma_3^n + b_4\gamma_4^n \\ &= \frac{1}{4}\rho_1^n - \frac{k}{12}\rho_2^n + \frac{k^2}{48}\rho_3^n - \frac{k^3}{24 \cdot 16}\rho_4^n + \frac{1}{3}\gamma_3^n + \frac{1}{6}\gamma_4^n. \end{aligned}$$

Therefore, in view of (4.101), we have

$$\varepsilon^{n+1} = \varepsilon^n - \frac{k}{2}\rho_1^n + \frac{k^2}{8}\rho_2^n - \frac{k^3}{48}\rho_3^n + \frac{k^4}{24 \cdot 16}\rho_4^n - k\gamma_5^n + \delta_1^n, \quad (4.134)$$

where

$$\gamma_5^n = \frac{1}{6}(\gamma_1^n + \gamma_4^n) + \frac{1}{3}(\gamma_2^n + \gamma_3^n),$$

satisfies, in view of (4.106), (4.114), (4.123) and (4.133), the inequality

$$\|\gamma_5^n\| \leq C_\lambda(\|\varepsilon^n\| + \|e^n\|_1), \quad (4.135)$$

for $n \leq n^*$. We begin now the estimation of

$$\sigma^n := \varepsilon^n - \frac{k}{2}\rho_1^n + \frac{k^2}{8}\rho_2^n - \frac{k^3}{48}\rho_3^n + \frac{k^4}{24 \cdot 16}\rho_4^n. \quad (4.136)$$

We have

$$\begin{aligned} \|\sigma^n\|^2 &= \|\varepsilon^n\|^2 - k(\varepsilon^n, \rho_1^n) + \frac{k^2}{4}[\|\rho_1^n\|^2 + (\varepsilon^n, \rho_2^n)] - \frac{k^3}{24}[(\varepsilon^n, \rho_3^n) + 3(\rho_1^n, \rho_2^n)] \\ &\quad + \frac{k^4}{64}[\|\rho_2^n\|^2 + \frac{1}{3}(\varepsilon^n, \rho_4^n) + \frac{4}{3}(\rho_1^n, \rho_3^n)] - \frac{k^5}{24 \cdot 16}[(\rho_1^n, \rho_4^n) + 2(\rho_2^n, \rho_3^n)] \\ &\quad + \frac{k^6}{48^2}[\|\rho_3^n\|^2 + \frac{3}{2}(\rho_2^n, \rho_4^n)] - \frac{k^7}{24^2 \cdot 16}(\rho_3^n, \rho_4^n) + \frac{k^8}{24^2 \cdot 16^2}\|\rho_4^n\|^2, \end{aligned}$$

or

$$\|\sigma^n\|^2 = \|\varepsilon^n\|^2 - kf_1^n + \frac{k^2}{4}f_2^n - \frac{k^3}{24}f_3^n + \frac{k^4}{64}f_4^n - \frac{k^5}{24 \cdot 16}f_5^n + \frac{k^6}{48^2}f_6^n - \frac{k^7}{24^2 \cdot 16}f_7^n + \frac{k^8}{24^2 \cdot 16^2}\|\rho_4^n\|^2, \quad (4.137)$$

where the numbers f_1^n, \dots, f_7^n have been defined in the obvious manner. By (4.104) we have

$$f_1^n = (\varepsilon^n, \rho_1^n) = -(U^n \varepsilon_x^n, \varepsilon^n) = \frac{1}{2}(U_x^n \varepsilon^n, \varepsilon^n),$$

and therefore

$$|f_1^n| \leq C\|\varepsilon^n\|^2. \quad (4.138)$$

By (4.113) and (4.104)

$$(\varepsilon^n, \rho_2^n) = -(U^n \varepsilon_x^n, \rho_1^n) = -((U^n \varepsilon^n)_x, \rho_1^n) + (U_x^n \varepsilon^n, \rho_1^n) = -\|\rho_1^n\|^2 + (U_x^n \varepsilon^n, \rho_1^n).$$

Since

$$f_2^n = \|\rho_1^n\|^2 + (\varepsilon^n, \rho_2^n),$$

we conclude that

$$f_2^n = (U_x^n \varepsilon^n, \rho_1^n) \leq \frac{C}{h}\|\varepsilon^n\|^2. \quad (4.139)$$

Now

$$(\rho_1^n, \rho_2^n) = \frac{1}{2}(U_x^n \rho_1^n, \rho_1^n).$$

In addition, using (4.122), (4.104), and (4.113) we have

$$(\varepsilon^n, \rho_3^n) = -(U^n \varepsilon_x^n, \rho_2^n) = -(\rho_1^n, \rho_2^n) + (U_x^n \varepsilon^n, \rho_2^n) = -\frac{1}{2}(U_x^n \rho_1^n, \rho_1^n) + (U_x^n \varepsilon^n, \rho_2^n).$$

Hence, since

$$f_3^n = (\varepsilon^n, \rho_3^n) + 3(\rho_1^n, \rho_2^n),$$

we have

$$f_3^n = (U_x^n \rho_1^n, \rho_1^n) + (U_x^n \varepsilon^n, \rho_2^n),$$

from which

$$|f_3^n| \leq \frac{C}{h^2}\|\varepsilon^n\|^2. \quad (4.140)$$

Now

$$(\rho_1^n, \rho_3^n) = -(U^n \rho_{1x}^n, \rho_2^n) = -((U^n \rho_1^n)_x, \rho_2^n) + (U_x^n \rho_1^n, \rho_2^n) = -\|\rho_2^n\|^2 + (U_x^n \rho_1^n, \rho_2^n),$$

and by (4.132)

$$\begin{aligned} (\varepsilon^n, \rho_4^n) &= -(U^n \varepsilon_x^n, \rho_3^n) = -((U^n \varepsilon^n)_x, \rho_3^n) + (U_x^n \varepsilon^n, \rho_3^n) = -(\rho_1^n, \rho_3^n) + (U_x^n \varepsilon^n, \rho_3^n) \\ &= \|\rho_2^n\|^2 - (U_x^n \rho_1^n, \rho_2^n) + (U_x^n \varepsilon^n, \rho_3^n). \end{aligned}$$

So, since

$$f_4^n = \|\rho_2^n\|^2 + \frac{1}{3}(\varepsilon^n, \rho_4^n) + \frac{4}{3}(\rho_1^n, \rho_3^n),$$

we have

$$f_4^n = (U_x^n \rho_1^n, \rho_2^n) + \frac{1}{3}(U_x^n \varepsilon^n, \rho_3^n),$$

and thus

$$|f_4^n| \leq \frac{C}{h^3}\|\varepsilon^n\|^2. \quad (4.141)$$

We also have

$$(\rho_2^n, \rho_3^n) = \frac{1}{2}(U_x^n \rho_2^n, \rho_2^n),$$

and

$$\begin{aligned}(\rho_1^n, \rho_4^n) &= -(U^n \rho_{1x}^n, \rho_3^n) = -((U^n \rho_1^n)_x, \rho_3^n) + (U_x^n \rho_1^n, \rho_3^n) = -(\rho_2^n, \rho_3^n) + (U_x^n \rho_1^n, \rho_3^n) \\ &= -\frac{1}{2}(U_x^n \rho_2^n, \rho_2^n) + (U_x^n \rho_1^n, \rho_3^n).\end{aligned}$$

Since

$$f_5^n = (\rho_1^n, \rho_4^n) + 2(\rho_2^n, \rho_3^n),$$

we have

$$f_5^n = \frac{1}{2}(U_x^n \rho_2^n, \rho_2^n) + (U_x^n \rho_1^n, \rho_3^n),$$

and therefore

$$|f_5^n| \leq \frac{C}{h^4} \|\varepsilon^n\|^2. \quad (4.142)$$

Similarly,

$$(\rho_2^n, \rho_4^n) = -(U^n \rho_{2x}^n, \rho_3^n) = -((U^n \rho_2^n)_x, \rho_3^n) + (U_x^n \rho_2^n, \rho_3^n) = -\|\rho_3^n\|^2 + (U_x^n \rho_2^n, \rho_3^n).$$

Since

$$f_6^n = \|\rho_3^n\|^2 + \frac{3}{2}(\rho_2^n, \rho_4^n),$$

we have

$$f_6^n = \frac{3}{2}(U_x^n \rho_2^n, \rho_3^n) - \frac{1}{2}\|\rho_3^n\|^2 \leq \frac{C}{h^5} \|\varepsilon^n\|^2 - \frac{1}{2}\|\rho_3^n\|^2. \quad (4.143)$$

Finally,

$$f_7^n = (\rho_3^n, \rho_4^n) = \frac{1}{2}(U_x^n \rho_3^n, \rho_3^n),$$

and thus

$$|f_7^n| \leq \frac{C}{h^6} \|\varepsilon^n\|^2. \quad (4.144)$$

Now, from the inequalities (4.138)-(4.144) and the fact that

$$\|\rho_4^n\| \leq \frac{\tilde{C}}{h} \|\rho_3^n\|,$$

where \tilde{C} is a constant independent of k and h , we obtain in (4.137)

$$\|\sigma^n\|^2 \leq (1 + C_\lambda k) \|\varepsilon^n\|^2 + \frac{k^6}{2 \cdot 48^2} \left(\frac{\tilde{C} \lambda^2}{32} - 1 \right) \|\rho_3^n\|^2.$$

Thus, for $\lambda \leq \lambda_0 = \sqrt{32/\tilde{C}}$ we will have the following estimate of $\|\sigma^n\|$

$$\|\sigma^n\| \leq (1 + C_\lambda k) \|\varepsilon^n\|.$$

Therefore, from (4.134), (4.136) and (4.135), we finally obtain

$$\|\varepsilon^{n+1}\| \leq \|\varepsilon^n\| + C_\lambda k (\|\varepsilon^n\| + \|e^n\|_1) + \|\delta_1^n\|,$$

for $n = 0, 1, \dots, n^*$. From this inequality, (4.129) and Lemma 4.5 we obtain for $n \leq n^*$

$$\|\varepsilon^{n+1}\| + \|e^{n+1}\|_1 \leq (1 + C_\lambda k) (\|\varepsilon^n\| + \|e^n\|_1) + Ck(k^4 + h^{3.5} \sqrt{\ln 1/h}).$$

Therefore, by Gronwall's Lemma, and (4.69) we see that

$$\|\varepsilon^n\| + \|e^n\|_1 \leq C_\lambda (k^4 + h^{3.5} \sqrt{\ln 1/h}),$$

for $n = 0, 1, \dots, n^*$. Hence, if k and h are sufficiently small, the maximality of n^* contradicted and we may take $n^* = M - 1$. The conclusion of the proposition follows. \square

Remark 4.5. Arguing as in Remark 4.1 we have again $\|U_h^n - u(t^n)\|_\infty = O(k^4 + h^{3.5} \sqrt{\ln 1/h})$.

Remark 4.6. The analogous results of Lemma 4.5 of Proposition 4.3 hold for the system (CB) as well.

5. A NON-STANDARD GALERKIN METHOD

In this section we analyze a semidiscrete Galerkin approximation of (CB) and (SCB) which is non-standard in the sense that it approximates η by piecewise linear, continuous functions and u by piecewise quadratic, C_0^1 functions, defined on the same mesh. (The method may be generalized using the analogous pairs of higher-order splines but here we confine ourselves to the low-order case.) In the analysis, we consider only the uniform mesh case, where a series of superaccuracy results for the error of the L^2 -projection onto the space of piecewise linear continuous functions affords proving optimal-order error estimates.

5.1. Preliminaries and the basic results. Using the notation of paragraph 2.2, in addition to S_h^2 we let

$$S_{h,0}^3 := \{\phi \in C^1 : \phi|_{[x_j, x_{j+1}]} \in \mathbb{P}_2, \ 1 \leq j \leq N, \ \phi(0) = \phi(1) = 0\}.$$

In this section we shall denote by $a(\cdot, \cdot)$ the usual bilinear form $a(\psi, \chi) = (\psi, \chi) + \frac{1}{3}(\psi', \chi')$ defined for $\psi, \chi \in S_{h,0}^3$ and let $R_h : H_0^1 \rightarrow S_{h,0}^3$ be the elliptic projection operator defined for $v \in H_0^1$ by $((R_h v)', \chi') = (v', \chi') \ \forall \chi \in S_{h,0}^3$. The system (SCB) is discretized as follows: Seek $\eta_h : [0, T] \rightarrow S_h^2$, $u_h : [0, T] \rightarrow S_{h,0}^3$ such that for $0 \leq t \leq T$

$$(\eta_{ht}, \phi) + (u_{hx}, \phi) + \frac{1}{2}((\eta_h u_h)_x, \phi) = 0, \quad \forall \phi \in S_h^2 \quad (5.1)$$

$$a(u_{ht}, \chi) + (\eta_{hx}, \chi) + \frac{3}{2}(u_h u_{hx}, \chi) + \frac{1}{2}(\eta_h \eta_{hx}, \chi) = 0, \quad \forall \chi \in S_{h,0}^3 \quad (5.2)$$

and

$$\eta_h(0) = P\eta_0, \quad u_h(0) = R_h u_0, \quad (5.3)$$

where $P : L^2 \rightarrow S_h^2$ is the L^2 -projection operator onto S_h^2 . (We will treat only the case of (SCB), that of (CB) being similar.) The o.d.e. initial-value problem represented by (5.1)-(5.3) has a unique solution for any $T < \infty$. In fact, taking $\phi = \eta_h$ in (5.1) and $\chi = u_h$ in (5.2) and adding the resulting equations, we obtain for $t \geq 0$

$$\|\eta_h(t)\|^2 + \|u_h(t)\|_1^2 = \|\eta_h(0)\|^2 + \|u_h(0)\|_1^2, \quad (5.4)$$

and the desired conclusion follows by standard o.d.e. theory.

The plan of the section is as follows: We will start by stating the preliminary Lemmas 5.1-5.4 with the aid of which we will prove our main error estimate, Theorem 5.1. The proofs of the Lemmas depend on some superaccuracy estimates of the L^2 projection error $v - Pv$ for smooth functions v ; these are proved in §5.2. (Lemmas 5.5-5.10). Recall first the following standard estimates, [19],[30]. Suppose that $\eta \in C^2$ and $u \in C_0^3$. Then, if $\rho = \eta - P\eta$, $\sigma = u - R_h u$, we have

$$\|\rho\| + h\|\rho\|_1 \leq Ch^2, \quad \|\sigma\| + h\|\sigma\|_1 \leq Ch^3, \quad \|\sigma\|_\infty + h\|\rho\|_\infty \leq Ch^3.$$

Lemma 5.1. *Let $\eta \in C^4$, $v \in C_0^2$. If $\rho = \eta - P\eta$ and $\zeta \in S_h^2$ is such that*

$$(\zeta, \phi) = ((v\rho)', \phi) \quad \forall \phi \in S_h^2,$$

then $\|\zeta\| \leq Ch^3$.

Proof. The result follows from Lemma 5.9 and Lemma 2.1(ii). □

Lemma 5.2. *Let $\eta \in C^2$ and $v \in C_0^1$. If $\rho = \eta - P\eta$ and $\zeta \in S_h^2$ is such that*

$$(\zeta, \phi) = (v\rho, \phi) \quad \forall \phi \in S_h^2,$$

then $\|\zeta\| \leq Ch^3$.

Proof. The result follows from Lemma 5.10 and Lemma 2.1(ii). □

Lemma 5.3. *Let $\eta \in C^2$ and $u \in C_0^3$. If $\rho = \eta - P\eta$, $\sigma = u - R_h u$, then*

- (i) $(\rho', \psi) = 0 \quad \forall \psi \in S_{h,0}^3$
- (ii) $(\sigma', \phi) = 0 \quad \forall \phi \in S_h^2$.

Proof. The identity (i) follows from the observation that $(\rho', \psi) = (\rho, \psi') = 0$, since $\psi \in S_{h,0}^3$ implies $\psi' \in S_h^2$. In order to prove (ii), we use an argument similar to that of Wahlbin, [30,p.5]. Let $\{\phi_i\}_{i=1}^{N+1}$ be the usual hat function basis of S_h^2 associated with the uniform mesh $x_i = (i-1)h$, $1 \leq i \leq N+1$, i.e. with the property that $\phi_i(x_j) = \delta_{ij}$. Then, the function ψ_i defined for $1 \leq i \leq N+1$ on $[0, 1]$ by $\psi_i(x) = \int_0^x \phi_i d\xi - x \int_0^1 \phi_i d\xi$ belongs to $S_{h,0}^3$ and satisfies $\psi_i' = \phi_i - \int_0^1 \phi_i d\xi$. Since $(\sigma', 1) = 0$, it follows that $(\sigma', \phi_i) = (\sigma', \psi_i') = 0$. Hence, $(\sigma', \phi) = 0$ for any $\phi \in S_h^2$ and the proof of (ii) is completed. (Since $Pu' = (R_h u)'$ as it may be easily established, (ii) implies that $\sigma' = u' - Pu'$.) □

Lemma 5.4. *Let $\eta \in C^1$ and $u \in C_0^3$. If $\sigma = u - R_h u$ and $\zeta \in S_h^2$ is such that*

$$(\zeta, \phi) = ((\eta\sigma)', \phi) \quad \forall \phi \in S_h^2,$$

then $\|\zeta\| \leq Ch^3$.

Proof. Consider $\gamma_i = ((\eta\sigma)', \phi_i)$. Then, by the final remark in the proof of Lemma 5.3,

$$\gamma_i = (\eta'\sigma, \phi_i) + (\eta\sigma', \phi_i) = (\eta'\sigma, \phi_i) + (\eta(u' - Pu'), \phi_i).$$

Now, since $\|\sigma\|_\infty \leq Ch^3$, we have that $|(\eta'\sigma, \phi_i)| \leq C\|\sigma\|_\infty \int_0^1 \phi_i dx \leq Ch^4$. In addition, by Lemma 5.10 we obtain that $|(\eta(u' - Pu'), \phi_i)| \leq Ch^4$. We conclude that $|\gamma| = (\sum_{i=1}^{N+1} \gamma_i^2)^{1/2} \leq Ch^{3.5}$, from which the conclusion of the Lemma follows in view of Lemma 2.1(ii). \square

With the aid of these lemmas we may establish the basic result of this section:

Theorem 5.1. *Suppose that the solution (η, u) of (SCB) is such that $\eta \in C(0, T; C^4)$, $\eta_t \in C(0, T; C^2)$, $u, u_t \in C(0, T; C_0^3)$ and let (η_h, u_h) be the solution of the semidiscrete problem (5.1)-(5.3). Then*

$$(i) \quad \max_{0 \leq t \leq T} (\|u(t) - u_h(t)\| + h\|\eta(t) - \eta_h(t)\|) \leq Ch^3,$$

$$(ii) \quad \max_{0 \leq t \leq T} (\|u(t) - u_h(t)\|_\infty + h\|\eta(t) - \eta_h(t)\|_\infty) \leq Ch^3.$$

Proof. Let $\rho = \eta - P\eta$, $\theta = P\eta - \eta_h$, $\sigma = u - R_h u$, and $\xi = R_h u - u_h$. Then $\eta - \eta_h = \rho + \theta$, $u - u_h = \sigma + \xi$ and, by Lemma 5.3

$$(\theta_t, \phi) + (\xi_x, \phi) + \frac{1}{2}((\eta u - \eta_h u_h)_x, \phi) = 0 \quad \forall \phi \in S_h^2, \quad (5.5)$$

$$a(\xi_t, \psi) + (\theta_x, \psi) + \frac{3}{2}(uu_x - u_h u_{hx}, \psi) + \frac{1}{2}(\eta \eta_x - \eta_h \eta_{hx}, \psi) = -(\sigma_t, \psi) \quad \forall \psi \in S_{h,0}^3, \quad (5.6)$$

for $t \in [0, T]$. Since

$$\eta u - \eta_h u_h = \eta(u - u_h) + u(\eta - \eta_h) - (\eta - \eta_h)(u - u_h) = \eta(\sigma + \xi) + u(\rho + \theta) - (\rho + \theta)(\sigma + \xi),$$

it follows that

$$\eta u - \eta_h u_h = \eta\sigma + u\rho - \theta\xi + f,$$

where $f = \eta\xi + u\theta - \rho\sigma - \rho\xi - \theta\sigma$. In addition,

$$\begin{aligned} \eta \eta_x - \eta_h \eta_{hx} &= \eta(\eta_x - \eta_{hx}) + \eta_x(\eta - \eta_h) - (\eta - \eta_h)(\eta_x - \eta_{hx}) \\ &= \eta(\rho_x + \theta_x) + \eta_x(\rho + \theta) - (\rho + \theta)(\rho_x + \theta_x) \\ &= (\eta\rho)_x + (\eta\theta)_x - (\rho\theta)_x - \rho\rho_x - \theta\theta_x, \end{aligned}$$

whence

$$\eta \eta_x - \eta_h \eta_{hx} = (\eta\rho)_x - \theta\theta_x + g_x,$$

where $g = \eta\theta - \rho\theta - \frac{1}{2}\rho^2$. Finally,

$$\begin{aligned} uu_x - u_h u_{hx} &= u(u_x - u_{hx}) + u_x(u - u_h) - (u - u_h)(u_x - u_{hx}) \\ &= u(\sigma_x + \xi_x) + u_x(\sigma + \xi) - (\sigma + \xi)(\sigma_x + \xi_x) \\ &= (u\sigma)_x + (u\xi)_x - (\sigma\xi)_x - \sigma\sigma_x - \xi\xi_x, \end{aligned}$$

i.e.

$$uu_x - u_h u_{hx} = \hat{f}_x,$$

where $\hat{f} = u\sigma + u\xi - \sigma\xi - \frac{1}{2}\sigma^2 - \frac{1}{2}\xi^2$. Thus we rewrite (5.5) and (5.6) as

$$(\theta_t, \phi) + (\xi_x, \phi) + \frac{1}{2}((\eta\sigma)_x, \phi) + \frac{1}{2}((u\rho)_x, \phi) - \frac{1}{2}((\theta\xi)_x, \phi) + \frac{1}{2}(f_x, \phi) = 0 \quad \forall \phi \in S_h^2,$$

$$a(\xi_t, \psi) + (\theta_x, \psi) + \frac{3}{2}(\hat{f}_x, \psi) + \frac{1}{2}((\eta\rho)_x, \psi) - \frac{1}{2}((\theta\theta)_x, \psi) + \frac{1}{2}(g_x, \psi) = -(\sigma_t, \psi) \quad \forall \psi \in S_{h,0}^3.$$

Putting $\phi = \theta$, $\chi = \xi$ in the above and adding the resulting equations, we have for $0 \leq t \leq T$

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (\|\theta\|^2 + \|\xi\|_1^2) + \frac{1}{2}((\eta\sigma)_x, \theta) + \frac{1}{2}((u\rho)_x, \theta) + \frac{1}{2}(f_x, \theta) \\ + \frac{1}{2}((\eta\rho)_x, \xi) + \frac{3}{2}(\hat{f}_x, \xi) + \frac{1}{2}(g_x, \xi) = -(\sigma_t, \xi). \end{aligned} \quad (5.7)$$

We estimate the various terms in (5.7) as follows. By Lemma 5.4 we have

$$|((\eta\sigma)_x, \theta)| \leq Ch^3 \|\theta\|,$$

and by Lemma 5.1

$$|((u\rho)_x, \theta)| \leq Ch^3 \|\theta\|.$$

Using integration by parts and the standard estimates for ρ and σ we also have

$$|(f_x, \theta)| \leq C\|\xi\|_1\|\theta\| + Ch^4\|\theta\| + C\|\theta\|^2.$$

In addition, from Lemma 5.2, since $\xi_x \in S_h^2$

$$|((\eta\rho)_x, \xi)| = |(\eta\rho, \xi_x)| \leq Ch^3\|\xi\|_1.$$

Finally, it follows from integration by parts and standard estimates for σ that

$$|(\hat{f}_x, \xi)| = |(\hat{f}, \xi_x)| \leq Ch^3\|\xi\|_1 + C\|\xi\|^2,$$

and by standard estimates for ρ that

$$|(g_x, \xi)| = |(g, \xi_x)| \leq C\|\theta\|\|\xi\|_1 + Ch^4\|\xi\|_1.$$

Hence, from (5.7) and these estimates we deduce for $0 \leq t \leq T$

$$\frac{1}{2} \frac{d}{dt} (\|\theta\|^2 + \|\xi\|_1^2) \leq C(h^6 + \|\theta\|^2 + \|\xi\|_1^2),$$

and, consequently, by Gronwall's Lemma and (5.3) that

$$\|\theta\| + \|\xi\|_1 \leq Ch^3, \quad 0 \leq t \leq T,$$

and the conclusions of the Theorem follow in view of the approximation properties of S_h^2 and $S_{h,0}^3$, Sobolev's theorem and the L^∞ - L^2 inverse inequality in S_h^2 . \square

Remark 5.1. In view of Theorems 1.3.1 and 1.3.2 of [30] the conclusions of Theorem 5.1 remain valid if in place of $R_h u_0$ we use the elliptic projection of u_0 that is induced by the bilinear form $a(\cdot, \cdot)$ as was done in previous sections.

5.2. Superaccuracy estimates for the error of the L^2 projection. The proofs of Lemmas 5.1, 5.2, and 5.4 depend on some superaccuracy properties of the error of the L^2 projection onto S_h^2 of a smooth function, in the case of uniform mesh.

Suppose that $\eta \in C^3$, $u \in C_0^2$, and let P be the L^2 projection onto S_h^2 and $\rho = \eta - P\eta$. To estimate $(u\rho, \phi'_i)$, where $\{\phi_i\}$ is the standard basis of S_h^2 , it suffices to estimate the integrals

$$\int_{I_i} u\rho dx,$$

where $I_i = (x_i, x_{i+1})$, $i = 1, 2, \dots, N$. Putting $x_{i+1/2} = (x_i + x_{i+1})/2$, we will have

$$\int_{I_i} u\rho dx = u(x_{i+1/2}) \int_{I_i} \rho dx + u'(x_{i+1/2}) \int_{I_i} (x - x_{i+1/2})\rho(x) dx + \frac{1}{2} \int_{I_i} (x - x_{i+1/2})^2 \rho(x) u''(y_i) dx,$$

for some $y_i = y_i(x) \in I_i$. Since $\|\rho\|_\infty = O(h^2)$, the third integral in the right-hand side is of $O(h^5)$. Now, if $x \in I_i$

$$\rho(x) = \rho(x_{i+1/2}) + (x - x_{i+1/2})\rho'(x_{i+1/2}) + \frac{1}{2}(x - x_{i+1/2})^2 \eta''(x_{i+1/2}) + \frac{1}{6}(x - x_{i+1/2})^3 \eta'''(\tau_i),$$

for some $\tau_i = \tau_i(x) \in I_i$. Hence

$$\int_{I_i} (x - x_{i+1/2})\rho(x) dx = \rho'(x_{i+1/2}) \int_{I_i} (x - x_{i+1/2})^2 dx + O(h^5) = \frac{h^3}{12} \rho'(x_{i+1/2}) + O(h^5).$$

Thus

$$\int_{I_i} u\rho dx = u(x_{i+1/2}) \int_{I_i} \rho dx + \frac{h^3}{12} \rho'(x_{i+1/2}) u'(x_{i+1/2}) + O(h^5),$$

and, consequently, we must estimate the integral of ρ on I_i and the derivative of ρ at the midpoint of I_i . Both of these quantities turn out to be superaccurate as we prove by elementary techniques in the following two lemmas:

Lemma 5.5. *Let $\eta \in C^3$. Then*

$$\max_{1 \leq i \leq N} |\rho'(x_{i+1/2})| \leq Ch^2.$$

Proof. For $x \in I_i$ we have

$$\rho'(x) = \eta'(x) - (P\eta)'(x) = \eta'(x) - \frac{c_{i+1} - c_i}{h},$$

i.e.

$$\rho'(x_{i+1/2}) = \eta'(x_{i+1/2}) - \frac{c_{i+1} - c_i}{h}, \quad 1 \leq i \leq N, \quad (5.8)$$

where c_i is the i th component of the solution c of the linear system $Gc = b$, where $G_{ij} = (\phi_j, \phi_i)$, $1 \leq i, j \leq N+1$ and $b_i = (\eta, \phi_i)$, $i = 1, 2, \dots, N+1$. The equations of the system may be written explicitly as

$$\begin{aligned} 2c_1 + c_2 &= 6b_1/h, \\ c_1 + 4c_2 + c_3 &= 6b_2/h, \\ c_{i-1} + 4c_i + c_{i+1} &= 6b_i/h, \quad i = 3, 4, \dots, N-1, \\ c_{N-1} + 4c_N + c_{N+1} &= 6b_N/h, \\ c_N + 2c_{N+1} &= 6b_{N+1}/h. \end{aligned} \quad (5.9)$$

Hence

$$\begin{aligned} 3(c_2 - c_1) + (c_3 - c_2) &= 6(b_2 - 2b_1)/h, \\ (c_i - c_{i-1}) + 4(c_{i+1} - c_i) + (c_{i+2} - c_{i+1}) &= 6(b_{i+1} - b_i)/h, \quad 2 \leq i \leq N-1 \end{aligned}$$

and

$$(c_N - c_{N-1}) + 3(c_{N+1} - c_N) = 6(2b_{N+1} - b_N)/h.$$

Therefore we obtain the following equations for the differences $(c_{i+1} - c_i)/h$:

$$\begin{aligned} 3(c_2 - c_1)/h + (c_3 - c_2)/h &= 6(b_2 - 2b_1)/h^2, \\ (c_i - c_{i-1})/h + 4(c_{i+1} - c_i)/h + (c_{i+2} - c_{i+1})/h &= 6(b_{i+1} - b_i)/h^2, \quad i = 2, 3, \dots, N-1, \\ (c_N - c_{N-1})/h + 3(c_{N+1} - c_N)/h &= 6(2b_{N+1} - b_N)/h^2. \end{aligned}$$

Hence, if we put $\varepsilon'_i := \rho'(x_{i+1/2})$, $i = 1, 2, \dots, N$, then (5.8) implies that $\varepsilon' = (\varepsilon'_i)$ is the solution of the linear system

$$A\varepsilon' = r', \quad (5.10)$$

where A is the $N \times N$ tridiagonal matrix with $A_{11} = A_{NN} = 3$, $A_{ii} = 4$, $2 \leq i \leq N-1$, and $A_{ij} = 1$ if $|i - j| = 1$, and $r' = (r'_1, r'_2, \dots, r'_N)^T$, where

$$\begin{aligned} r'_1 &= 3\eta'(x_{3/2}) + \eta'(x_{5/2}) - 6(b_2 - 2b_1)/h^2, \\ r'_i &= \eta'(x_{i-1/2}) + 4\eta'(x_{i+1/2}) + \eta'(x_{i+3/2}) - 6(b_{i+1} - b_i)/h^2, \quad i = 2, 3, \dots, N-1, \\ r'_N &= \eta'(x_{N-1/2}) + 3\eta'(x_{N+1/2}) - 6(2b_{N+1} - b_N)/h^2. \end{aligned} \quad (5.11)$$

We will now show that there exists a constant C depending only on $\|\eta'''\|_\infty$, such that

$$\max_{1 \leq i \leq N} |r'_i| \leq Ch^2. \quad (5.12)$$

To estimate r'_1 note that

$$\begin{aligned} b_2 - 2b_1 &= \int_{I_1 \cup I_2} \eta \phi_2 dx - 2 \int_{I_1} \eta \phi_1 dx = \frac{3}{h} \int_{I_1} (x - h) \eta dx - \frac{1}{h} \int_{I_2} (x - h) \eta dx + \int_{I_1 \cup I_2} \eta dx \\ &=: \frac{3}{h} J_1 - \frac{1}{h} J_2 + J_3. \end{aligned}$$

Using Taylor's theorem we have

$$\begin{aligned} J_1 &= \int_{I_1} (x - h) \eta dx = -\frac{h^2}{2} \eta(h) + \frac{h^3}{3} \eta'(h) - \frac{h^4}{8} \eta''(h) + O(h^5), \\ J_2 &= \int_{I_2} (x - h) \eta dx = \frac{h^2}{2} \eta(h) + \frac{h^3}{3} \eta'(h) + \frac{h^4}{8} \eta''(h) + O(h^5), \\ J_3 &= \int_{I_1 \cup I_2} \eta dx = 2h\eta(h) + \frac{h^3}{3} \eta''(h) + O(h^4), \end{aligned}$$

and thus

$$b_2 - 2b_1 = \frac{2h^2}{3}\eta'(h) - \frac{h^3}{6}\eta''(h) + O(h^4).$$

In addition,

$$3\eta'(x_{3/2}) + \eta'(x_{5/2}) = 4\eta'(h) - h\eta''(h) + O(h^2).$$

Therefore, by (5.11), we see that $r'_1 = O(h^2)$. For r'_N we have

$$\begin{aligned} 2b_{N+1} - b_N &= 2 \int_{I_N} \eta \phi_{N+1} dx - \int_{I_{N-1} \cup I_N} \eta \phi_N dx \\ &= \frac{3}{h} \int_{I_N} (x - x_N) \eta dx - \frac{1}{h} \int_{I_{N-1}} (x - x_N) \eta dx - \int_{I_{N-1} \cup I_N} \eta dx \\ &=: \frac{3}{h} J_1 - \frac{1}{h} J_2 - J_3. \end{aligned}$$

By Taylor's theorem

$$\begin{aligned} J_1 &= \frac{h^2}{2}\eta(x_N) + \frac{h^3}{3}\eta'(x_N) + \frac{h^4}{8}\eta''(x_N) + O(h^5), \\ J_2 &= -\frac{h^2}{2}\eta(x_N) + \frac{h^3}{3}\eta'(x_N) - \frac{h^4}{8}\eta''(x_N) + O(h^5), \\ J_3 &= 2h\eta(x_N) + \frac{h^3}{3}\eta''(x_N) + O(h^4), \end{aligned}$$

and thus

$$2b_{N+1} - b_N = \frac{2h^2}{3}\eta'(x_N) + \frac{h^3}{6}\eta''(x_N) + O(h^4).$$

Hence, by (5.11)

$$\begin{aligned} r'_N &= \eta'(x_{N-1/2}) + 3\eta'(x_{N+1/2}) - \frac{6}{h^2}(2b_{N+1} - b_N) \\ &= 4\eta'(x_N) + h\eta''(x_N) - 4\eta'(x_N) - h\eta''(x_N) + O(h^2) = O(h^2). \end{aligned}$$

For r'_i , $2 \leq i \leq N-1$, we have

$$\begin{aligned} b_{i+1} - b_i &= \int_{I_i \cup I_{i+1}} \eta \phi_{i+1} dx - \int_{I_{i-1} \cup I_i} \eta \phi_i dx \\ &= \frac{2}{h} \int_{I_i} (x - x_{i+1/2}) \eta dx - \frac{1}{h} \int_{I_{i+1}} (x - x_{i+3/2}) \eta dx - \frac{1}{h} \int_{I_{i-1}} (x - x_{i-1/2}) \eta dx \\ &\quad + \frac{1}{2} \int_{I_{i+1}} \eta dx - \frac{1}{2} \int_{I_{i-1}} \eta dx =: \frac{2}{h} J_1 - \frac{1}{h} J_2 - \frac{1}{h} J_3 + \frac{1}{2} J_4 - \frac{1}{2} J_5. \end{aligned} \tag{5.13}$$

By Taylor's theorem we obtain

$$J_1 = \eta'(x_{i+1/2}) \int_{I_i} (x - x_{i+1/2})^2 dx + O(h^5) = \frac{h^3}{12}\eta'(x_{i+1/2}) + O(h^5). \tag{5.14}$$

Similarly

$$J_2 = \frac{h^3}{12}\eta'(x_{i+3/2}) + O(h^5), \tag{5.15}$$

$$J_3 = \frac{h^3}{12}\eta'(x_{i-1/2}) + O(h^5), \tag{5.16}$$

$$J_4 = h\eta(x_{i+3/2}) + \frac{h^3}{24}\eta''(x_{i+3/2}) + O(h^4),$$

$$J_5 = h\eta(x_{i-1/2}) + \frac{h^3}{24}\eta''(x_{i-1/2}) + O(h^4).$$

Since now

$$J_4 - J_5 = 2h^2\eta'(x_{i+1/2}) + O(h^4). \tag{5.17}$$

Substituting (5.14)-(5.17) into (5.13) we obtain

$$b_{i+1} - b_i = \frac{7h^2}{6}\eta'(x_{i+1/2}) - \frac{h^2}{12}\eta'(x_{i+3/2}) - \frac{h^2}{12}\eta'(x_{i-1/2}) + O(h^4),$$

and finally, by (5.11)

$$\begin{aligned} r'_i &= \eta'(x_{i-1/2}) + 4\eta'(x_{i+1/2}) + \eta'(x_{i+3/2}) - \frac{6}{h^2}(b_{i+1} - b_i) \\ &= \frac{3}{2}(\eta'(x_{i+3/2}) - 2\eta'(x_{i+1/2}) + \eta'(x_{i-1/2})) + O(h^2) = O(h^2). \end{aligned}$$

We conclude therefore that (5.12) holds with a constant $C = C(\|\eta'''\|_\infty)$. Writing the linear system (5.9) as $\frac{1}{4}A\varepsilon' = \frac{1}{4}r'$ with $\frac{1}{4}A = I - B$, where B is a $N \times N$ matrix with $\|B\|_\infty = 1/2$. Therefore, $\|(I - B)^{-1}\|_\infty \leq 2$ and consequently

$$\max_{1 \leq i \leq N} |\varepsilon'_i| \leq \frac{1}{2} \max_{1 \leq i \leq N} |r'_i| \leq Ch^2.$$

□

Lemma 5.6. *Let $\eta \in C^3$. Then for some constant $C = C(\|\eta'''\|_\infty)$ we have*

$$\max_{1 \leq i \leq N} \left| \int_{I_i} \rho dx \right| \leq Ch^4.$$

Proof. We have

$$\int_{I_i} \rho dx = \int_{I_i} \eta dx - \frac{h}{2}(c_{i+1} + c_i), \quad i = 1, 2, \dots, N,$$

where c_i is the i -th component of the solution c of the linear system $Gc = b$ with G and b as in the previous Lemma. From the equation (5.9) we see that

$$\begin{aligned} 5 \cdot \frac{h}{2}(c_1 + c_2) + \frac{h}{2}(c_2 + c_3) &= 3(2b_1 + b_2), \\ \frac{h}{2}(c_{i-1} + c_i) + 4 \cdot \frac{h}{2}(c_i + c_{i+1}) + \frac{h}{2}(c_{i+1} + c_{i+2}) &= 3(b_{i+1} + b_i), \quad 2 \leq i \leq N-1, \\ \frac{h}{2}(c_{N-1} + c_N) + 5 \cdot \frac{h}{2}(c_N + c_{N+1}) &= 3(b_N + 2b_{N+1}). \end{aligned}$$

Hence, if

$$\varepsilon_i := \int_{I_i} \rho dx, \quad i = 1, 2, \dots, N,$$

then $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_N)^T$ is the solution of the system

$$\Gamma \varepsilon = r, \tag{5.18}$$

where Γ is the tridiagonal $N \times N$ matrix with $\Gamma_{11} = \Gamma_{NN} = 5$, $\Gamma_{ii} = 4$, $i = 2, 3, \dots, N-1$, and $\Gamma_{ij} = 1$, when $|i - j| = 1$, and where $r = (r_1, r_2, \dots, r_N)^T$ with

$$\begin{aligned} r_1 &= 5 \int_{I_1} \eta dx + \int_{I_2} \eta dx - 3(2b_1 + b_2), \\ r_i &= \int_{I_{i-1}} \eta dx + 4 \int_{I_i} \eta dx + \int_{I_{i+1}} \eta dx - 3(b_{i+1} + b_i), \quad i = 2, 3, \dots, N-1, \\ r_N &= \int_{I_{N-1}} \eta dx + 5 \int_{I_N} \eta dx - 3(b_N + 2b_{N+1}). \end{aligned} \tag{5.19}$$

We will show that $r_i = O(h^4)$, for $i = 1, 2, \dots, N$.

For r_1 , note that

$$2b_1 + b_2 = 2 \int_{I_1} \eta \phi_1 dx + \int_{I_1 \cup I_2} \eta \phi_2 dx = \int_{I_1 \cup I_2} \eta dx - \frac{1}{h} \int_{I_1 \cup I_2} (x - h) \eta dx.$$

Hence, by Taylor's theorem

$$2b_1 + b_2 = 2h\eta(h) - \frac{2h^2}{3}\eta'(h) + \frac{h^3}{3}\eta''(h) + O(h^4). \tag{5.20}$$

Also

$$\begin{aligned} \int_{I_1} \eta dx &= h\eta(h) - \frac{h^2}{2}\eta'(h) + \frac{h^3}{6}\eta''(h) + O(h^4), \\ \int_{I_2} \eta dx &= h\eta(h) + \frac{h^2}{2}\eta'(h) + \frac{h^3}{6}\eta''(h) + O(h^4). \end{aligned}$$

From these relations and (5.19) and (5.20) we obtain

$$\begin{aligned} r_1 &= 5(h\eta(h) - \frac{h^2}{2}\eta'(h) + \frac{h^3}{6}\eta''(h)) + (h\eta(h) + \frac{h^2}{2}\eta'(h) + \frac{h^3}{6}\eta''(h)) \\ &\quad - 3(2h\eta(h) - \frac{2h^2}{3}\eta'(h) + \frac{h^3}{3}\eta''(h)) + O(h^4) = O(h^4). \end{aligned}$$

For r_N we observe

$$b_N + 2b_{N+1} = \int_{I_{N-1} \cup I_N} \eta \phi_N dx + 2 \int_{I_N} \eta \phi_{N+1} dx = \frac{1}{h} \int_{I_{N-1} \cup I_N} (x - x_N) \eta dx + \int_{I_{N-1} \cup I_N} \eta dx.$$

Hence, by Taylor's theorem

$$b_N + 2b_{N+1} = 2h\eta(x_N) + \frac{2h^2}{3}\eta'(x_N) + \frac{h^3}{3}\eta''(x_N) + O(h^4). \quad (5.21)$$

Also

$$\begin{aligned} \int_{I_{N-1}} \eta dx &= h\eta(x_N) - \frac{h^2}{2}\eta'(x_N) + \frac{h^3}{6}\eta''(x_N) + O(h^4), \\ \int_{I_N} \eta dx &= h\eta(x_N) + \frac{h^2}{2}\eta'(x_N) + \frac{h^3}{6}\eta''(x_N) + O(h^4). \end{aligned}$$

From these relations and (5.19) and (5.21) we see that

$$\begin{aligned} r_N &= (h\eta(x_N) - \frac{h^2}{2}\eta'(x_N) + \frac{h^3}{6}\eta''(x_N)) + 5(h\eta(x_N) + \frac{h^2}{2}\eta'(x_N) + \frac{h^3}{6}\eta''(x_N)) \\ &\quad - 3(2h\eta(x_N) + \frac{2h^2}{3}\eta'(x_N) + \frac{h^3}{3}\eta''(x_N)) + O(h^4) = O(h^4). \end{aligned}$$

For r_i , $i = 2, 3, \dots, N-1$, we have

$$\begin{aligned} b_i + b_{i+1} &= \int_{I_{i-1} \cup I_i} \eta \phi_i dx + \int_{I_i \cup I_{i+1}} \eta \phi_i dx \\ &= \frac{1}{h} \int_{I_{i-1}} (x - x_{i-1}) \eta dx + \int_{I_i} \eta dx + \frac{1}{h} \int_{I_{i+1}} (x_{i+2} - x) \eta dx. \end{aligned}$$

Thus, by (5.19) we have, for $2 \leq i \leq N-1$

$$r_i = \int_{I_{i-1} \cup I_i \cup I_{i+1}} \eta dx - \frac{3}{h} \int_{I_{i-1}} (x - x_{i-1}) \eta dx - \frac{3}{h} \int_{I_{i+1}} (x_{i+2} - x) \eta dx =: J_1 - \frac{3}{h} J_2 - \frac{3}{h} J_3. \quad (5.22)$$

By Taylor's theorem

$$\begin{aligned} J_2 &= \frac{h^2}{2}\eta(x_{i-1}) + \frac{h^3}{3}\eta'(x_{i-1}) + \frac{h^4}{8}\eta''(x_{i-1}) + O(h^5), \\ J_3 &= \frac{h^2}{2}\eta(x_{i+2}) - \frac{h^3}{3}\eta'(x_{i+2}) + \frac{h^4}{8}\eta''(x_{i+2}) + O(h^5), \end{aligned}$$

i.e.

$$\begin{aligned} J_2 + J_3 &= \frac{h^2}{2}(\eta(x_{i-1}) + \eta(x_{i+2})) + \frac{h^3}{3}(\eta'(x_{i-1}) - \eta'(x_{i+2})) + \frac{h^4}{8}(\eta''(x_{i-1}) + \eta''(x_{i+2})) + O(h^5) \\ &= h^2\eta(x_{i+1/2}) + \frac{3h^4}{8}\eta''(x_{i+1/2}) + O(h^5). \end{aligned}$$

In addition,

$$J_1 = 3h\eta(x_{i+1/2}) + \frac{1}{3} \frac{27h^3}{8}\eta''(x_{i+1/2}) + O(h^4).$$

Substituting in (5.22) and in (5.19) we see that for $2 \leq i \leq N-1$

$$r_i = 3h\eta(x_{i+1/2}) + \frac{9h^3}{8}\eta''(x_{i+1/2}) - 3h\eta(x_{i+1/2}) - \frac{9h^3}{8}\eta''(x_{i+1/2}) + O(h^4) = O(h^4).$$

Hence, there exists a constant C depending only on $\|\eta'''\|_\infty$ such that

$$\max_{1 \leq i \leq N} |r_i| \leq Ch^4. \quad (5.23)$$

From (5.18) we obtain that $\frac{1}{4}\Gamma\varepsilon = \frac{1}{4}r$, where $\frac{1}{4}\Gamma = I - E$, and E is a $N \times N$ matrix with $\|E\|_\infty = 1/2$. Hence $\|(I - E)^{-1}\|_\infty \leq 2$ and thus

$$\max_{1 \leq i \leq N} |\varepsilon_i| \leq \frac{1}{2} \max_{1 \leq i \leq N} |r_i| \leq Ch^4.$$

□

If we now assume that η is smoother, then, using a result by Demko, [17], (see also de Boor, [16]), we may show that $\int_{I_i} \rho dx = O(h^5)$ provided that the endpoints x_i, x_{i+1} of the interval I_i are at a distance of $O(h \ln 1/h)$ from the boundary of $[0, 1]$.

Lemma 5.7. *Let $\eta \in C^4$. Then, there exists a constant \tilde{C} such that if*

$$\text{dist}(x_i, 0) \geq \tilde{C}h \ln 1/h \quad \text{and} \quad \text{dist}(x_{i+1}, 1) \geq \tilde{C}h \ln 1/h,$$

then

$$\int_{I_i} \rho dx = O(h^5).$$

Proof. We will first show that due to our hypothesis of increased smoothness of η , we now have

$$r_i = O(h^5), \quad i = 2, 3, \dots, N-1,$$

where the r_i were defined in (5.19) and written in the form (5.22). Since $\eta \in C^4$, using Simpson's rule we approximate the integrals in (5.22) by

$$\begin{aligned} \int_{I_{i-1} \cup I_i \cup I_{i+1}} \eta dx &= \frac{3h}{6} (\eta(x_{i-1}) + 4\eta(x_{i+1/2}) + \eta(x_{i+2})) + O(h^5), \\ \int_{I_{i-1}} \frac{x-x_{i-1}}{h} \eta dx &= \frac{h}{6} (2\eta(x_{i-1/2}) + \eta(x_i)) + O(h^5), \\ \int_{I_{i+1}} \frac{x_{i+2}-x}{h} \eta dx &= \frac{h}{6} (\eta(x_{i+1}) + 2\eta(x_{i+3/2})) + O(h^5), \end{aligned}$$

yielding

$$r_i = \frac{h}{2} (\eta(x_{i-1}) - 2\eta(x_{i-1/2}) - \eta(x_i) + 4\eta(x_{i+1/2}) + \eta(x_{i+2}) - 2\eta(x_{i+3/2}) - \eta(x_{i+1})) + O(h^5).$$

Hence, using Taylor's theorem

$$\begin{aligned} r_i &= \frac{h}{2} \left(\frac{h^2}{4} \eta''(x_{i-1/2}) - 2\frac{h^2}{4} \eta''(x_{i+1/2}) + \frac{h^2}{4} \eta''(x_{i+3/2}) \right) + O(h^5) \\ &= \frac{h^3}{8} (\eta''(x_{i-1/2}) - 2\eta''(x_{i+1/2}) + \eta''(x_{i+3/2})) + O(h^5) = O(h^5). \end{aligned}$$

We conclude that for constants C_1, C_2 independent of h we have, in view of (5.23), that

$$|r_1| \leq C_1 h^4, \quad \max_{2 \leq i \leq N-1} |r_i| \leq C_2 h^5, \quad |r_N| \leq C_1 h^4. \quad (5.24)$$

We also recall from Lemma 5.6, cf. (5.18), that the $\varepsilon_i := \int_{I_i} \rho dx$, $1 \leq i \leq N$, form the solution of the system

$$\frac{1}{4} \Gamma \varepsilon = (I - E) \varepsilon = \frac{1}{4} r,$$

where Γ was defined after (5.18). Hence

$$\varepsilon_i = \sum_{j=1}^N (\Gamma^{-1})_{ij} r_j = (\Gamma^{-1})_{i1} r_1 + \sum_{j=2}^{N-1} (\Gamma^{-1})_{ij} r_j + (\Gamma^{-1})_{iN} r_N. \quad (5.25)$$

Now, using Proposition 2.1 of [17], we see that there is a constant C_3 such that

$$|(\Gamma^{-1})_{i1}| \leq C_3 2^{-|i-1|}, \quad \text{and} \quad |(\Gamma^{-1})_{iN}| \leq C_3 2^{-|i-N|}. \quad (5.26)$$

Therefore, if $\text{dist}(x_i, 0) \geq \frac{1}{\ln 2} h \ln 1/h$ and $\text{dist}(x_{i+1}, 1) \geq \frac{1}{\ln 2} h \ln 1/h$, there follows that

$$i-1 \geq \frac{1}{\ln 2} \ln 1/h \quad \text{and} \quad N-i \geq \frac{1}{\ln 2} \ln 1/h,$$

i.e.

$$2^{-|i-1|} \leq h \quad \text{and} \quad 2^{-|i-N|} \leq h,$$

and by (5.26)

$$|(\Gamma^{-1})_{i1}| \leq C_3 h \quad \text{and} \quad |(\Gamma^{-1})_{iN}| \leq C_3 h.$$

Hence, for the indices i for which

$$1 + \frac{1}{\ln 2} \ln 1/h \leq i \leq N - \frac{1}{\ln 2} \ln 1/h,$$

we will have from (5.25) and (5.24) that

$$\begin{aligned} |\varepsilon_i| &\leq C_3 h |r_1| + \sum_{j=2}^{N-1} |(\Gamma^{-1})_{ij}| |r_i| + C_3 h |r_N| \\ &\leq C_3 C_1 h^5 + \max_{1 \leq i \leq N} \sum_{j=1}^N |(\Gamma^{-1})_{ij}| C_2 h^5 + C_3 C_1 h^5 \leq C h^5. \end{aligned}$$

□

We now prove, as consequences of these results, several superaccurate estimates on weighted integrals of the error of the L^2 projection.

Lemma 5.8. *Let $\eta \in C^4$, $u \in C_0^2$, and $\rho = \eta - P\eta$. Then,*

$$\max_{1 \leq i \leq N} \left| \int_{I_i} u \rho dx \right| \leq C h^5 \ln 1/h.$$

Proof. If $1 \leq i \leq N$ we have for $t_i = t_i(x) \in I_i$

$$\int_{I_i} u \rho dx = u(x_{i+1/2}) \int_{I_i} \rho dx + u'(x_{i+1/2}) \int_{I_i} (x - x_{i+1/2}) \rho dx + \frac{1}{2} \int_{I_i} (x - x_{i+1/2})^2 \rho u''(t_i) dx.$$

The third integral of the right-hand side is of $O(h^5)$. For the second integral, by Taylor's theorem we have

$$\int_{I_i} (x - x_{i+1/2}) \rho dx = \frac{h^3}{12} \rho'(x_{i+1/2}) + O(h^5),$$

and using Lemma 5.5,

$$\int_{I_i} (x - x_{i+1/2}) \rho dx = O(h^5).$$

We conclude that

$$\int_{I_i} u \rho dx = u(x_{i+1/2}) \int_{I_i} \rho dx + O(h^5). \quad (5.27)$$

If now i is such that

$$\text{dist}(x_i, 0) \geq \tilde{C} h \ln 1/h, \quad \text{and} \quad \text{dist}(x_{i+1}, 1) \geq \tilde{C} h \ln 1/h,$$

where \tilde{C} is the constant in the statement of Lemma 5.7, then

$$\int_{I_i} u \rho dx = O(h^5). \quad (5.28)$$

If $\text{dist}(x_i, 0) \leq \tilde{C} h \ln 1/h$, we have

$$u(x_{i+1/2}) = u(x_i) + \frac{h}{2} u'(\tau_i), \quad u(x_i) = x_i u'(y_i),$$

for some $\tau_i \in (x_i, x_{i+1/2})$ and $y_i \in (0, x_i)$. Therefore

$$|u(x_{i+1/2})| \leq C h \ln 1/h,$$

and from Lemma 5.6 and (5.27) we have for such i that

$$\left| \int_{I_i} u \rho dx \right| \leq C h^5 \ln 1/h. \quad (5.29)$$

If finally $\text{dist}(x_{i+1}, 1) \leq \tilde{C} h \ln 1/h$, we have

$$u(x_{i+1/2}) = u(x_{i+1}) - \frac{h}{2} u'(\tilde{\tau}_i), \quad u(x_{i+1}) = (1 - x_{i+1}) u'(\tilde{y}_i),$$

for some $\tilde{\tau}_i \in (x_{i+1/2}, x_{i+1})$ and $\tilde{y}_i \in (x_{i+1}, 1)$. Hence

$$|u(x_{i+1/2})| \leq C h \ln 1/h,$$

and it follows again from Lemma 5.6 and (5.27) that in this case too

$$\left| \int_{I_i} u \rho dx \right| \leq C h^5 \ln 1/h. \quad (5.30)$$

The conclusion of the Lemma follows from (5.28)-(5.30). \square

Lemma 5.9. *Let $\eta \in C^4$, $u \in C_0^2$, $\rho = \eta - P\eta$, and*

$$\beta_i := (u\rho, \phi_i'), \quad i = 1, 2, \dots, N+1.$$

If $\beta = (\beta_1, \beta_2, \dots, \beta_{N+1})^T$ then

$$|\beta| \leq Ch^{3.5},$$

where $|\cdot|$ is the l_2 norm in \mathbb{R}^{N+1} .

Proof. We have

$$\begin{aligned} \beta_1 &= -\frac{1}{h} \int_{I_1} u\rho dx, \\ \beta_i &= \frac{1}{h} \int_{I_{i-1}} u\rho dx - \frac{1}{h} \int_{I_i} u\rho dx, \quad i = 2, 3, \dots, N, \\ \beta_{N+1} &= \frac{1}{h} \int_{I_N} u\rho dx. \end{aligned}$$

If $\text{dist}(x_{i-1}, 0) \geq \tilde{C}h \ln 1/h$ and $\text{dist}(x_{i+1}, 1) \geq \tilde{C}h \ln 1/h$, i.e. if $i \in K$, where K are the integers in the interval $[2 + \tilde{C} \ln 1/h, N - \tilde{C} \ln 1/h]$, then by (5.28)

$$|\beta_i| \leq Ch^4. \quad (5.31)$$

If on the other hand $\text{dist}(x_{i-1}, 0) \leq \tilde{C}h \ln 1/h$ or $\text{dist}(x_{i+1}, 1) \leq \tilde{C}h \ln 1/h$, i.e. if $i \in K_1$, where K_1 are the integers in $[1, 1 + \tilde{C} \ln 1/h] \cup [N + 1 - \tilde{C} \ln 1/h, N + 1]$, then, by (5.29) and (5.30)

$$|\beta_i| \leq Ch^4 \ln 1/h.$$

Therefore,

$$\begin{aligned} |\beta|^2 &= \sum_{i=1}^{N+1} |\beta_i|^2 = \sum_{i \in K_1} |\beta_i|^2 + \sum_{i \in K} |\beta_i|^2 \\ &\leq (1 + 2\tilde{C} \ln 1/h) Ch^8 (\ln 1/h)^2 + C(N - 1 - 2\tilde{C} \ln 1/h) h^8 \leq Ch^7. \end{aligned}$$

\square

Lemma 5.10. *Let $w \in C^2$, $v \in C^1$ and*

$$\beta_i = (v(w - Pw), \phi_i), \quad i = 1, 2, \dots, N+1.$$

If $\beta = (\beta_1, \beta_2, \dots, \beta_{N+1})^T$, then

$$|\beta| \leq Ch^{3.5},$$

where $|\cdot|$ is the l_2 norm in \mathbb{R}^{N+1} .

Proof. For $x \in I_1$ we have

$$v(x) = v(h) + O(h),$$

and since $\|w - Pw\|_\infty = O(h^2)$,

$$v(x)(w - Pw)(x) = v(h)(w - Pw)(x) + O(h^3),$$

and therefore

$$\beta_1 = O(h^4).$$

If $x \in I_{i-1} \cup I_i$, $i = 2, 3, \dots, N$, we have

$$v(x) = v(x_i) + O(h),$$

whence

$$v(x)(w - Pw)(x) = v(x_i)(w - Pw)(x) + O(h^3),$$

and so

$$\beta_i = O(h^4).$$

Similarly, $\beta_{N+1} = O(h^4)$, and finally $|\beta| = O(h^{3.5})$. \square

5.3. Numerical experiments. We considered first the (CB) system, that we discretized by the nonstandard method analyzed in the previous two sections using the subspaces S_h^2 and $S_{h,0}^3$ for approximating η and u , respectively. (We considered the nonhomogeneous system with a suitable right-hand side so that the solution is $\eta = \exp(2t)(\cos(\pi x) + x^2 + 2)$, $u = \exp(xt)(\sin(\pi x) + x^3 - x^2)$.) We integrated the semidiscrete system up to $T = 1$ by the fourth-order accurate explicit Runge-Kutta method analyzed in paragraph 4.3 with time step $k = h/10$; the temporal discretization error was negligible in comparison to the spatial error. Table

N	η	order	u	order
40	0.1250(-2)		0.4057(-5)	
60	0.5555(-3)	2.001	0.1199(-5)	3.008
80	0.3124(-3)	2.000	0.5051(-6)	3.004
100	0.1999(-3)	2.000	0.2585(-6)	3.002
120	0.1388(-3)	2.000	0.1495(-6)	3.001
140	0.1020(-3)	2.000	0.9416(-7)	3.001
160	0.7810(-4)	2.000	0.6307(-7)	3.001
180	0.6170(-4)	2.000	0.4429(-7)	3.001

N	η	order	u	order
40	0.3342(-2)		0.5955(-5)	
60	0.1485(-2)	2.000	0.1780(-5)	2.979
80	0.8357(-3)	1.999	0.7539(-6)	2.985
100	0.5349(-3)	2.000	0.3870(-6)	2.989
120	0.3714(-3)	2.000	0.2243(-6)	2.991
140	0.2729(-3)	2.000	0.1414(-6)	2.992
160	0.2089(-3)	2.000	0.9483(-7)	2.993
180	0.1651(-3)	2.000	0.6665(-7)	2.994

N	η	order	u	order
40	0.3872		0.1048(-2)	
60	0.2581	1.000	0.4652(-3)	2.002
80	0.1936	1.000	0.2616(-3)	2.001
100	0.1549	1.000	0.1674(-3)	2.001
120	0.1290	1.000	0.1162(-3)	2.000
140	0.1106	1.000	0.8540(-4)	2.000
160	0.9678(-1)	1.000	0.6538(-4)	2.000
180	0.8603(-1)	1.000	0.5166(-4)	2.000

TABLE 5.1. Errors and orders of convergence. (CB) system, nonstandard Galerkin semidiscretization, $\eta_h \in S_h^2$, $u_h \in S_{h,0}^3$, uniform mesh.

5.1 shows the resulting errors and orders of convergence in various norms. The optimal-order L^2 - and L^∞ rates of convergence of Theorem 5.1 are confirmed and the H^1 rates are as expected. We obtained the same rates for the (SCB) system. Table 5.2 shows the analogous errors and rates of convergence for the same

N	η	order	u	order
40	0.2214(-4)		0.1529(-6)	
60	0.6628(-5)	2.975	0.3044(-7)	3.981
80	0.2810(-5)	2.982	0.9670(-8)	3.986
100	0.1443(-5)	2.986	0.3970(-8)	3.989
120	0.8370(-6)	2.989	0.1918(-8)	3.991
140	0.5279(-6)	2.990	0.1036(-8)	3.992
160	0.3540(-6)	2.992	0.6081(-9)	3.993
180	0.2488(-6)	2.993	0.3799(-9)	3.994

N	η	order	u	order
40	0.4812(-4)		0.6323(-6)	
60	0.1422(-4)	3.006	0.1290(-6)	3.921
80	0.5993(-5)	3.005	0.4145(-7)	3.946
100	0.3066(-5)	3.004	0.1713(-7)	3.958
120	0.1773(-5)	3.003	0.8314(-8)	3.966
140	0.1116(-5)	3.003	0.4507(-8)	3.972
160	0.7474(-6)	3.003	0.2651(-8)	3.976
180	0.5248(-6)	3.002	0.1659(-8)	3.979

N	η	order	u	order
40	0.5999(-2)		0.3854(-4)	
60	0.2654(-2)	2.011	0.1152(-4)	2.978
80	0.1490(-2)	2.008	0.4884(-5)	2.984
100	0.9520(-3)	2.006	0.2509(-5)	2.988
120	0.6605(-3)	2.005	0.1454(-5)	2.990
140	0.4850(-3)	2.004	0.9168(-6)	2.991
160	0.3711(-3)	2.004	0.6148(-6)	2.992
180	0.2931(-3)	2.003	0.4321(-6)	2.993

TABLE 5.2. Errors and orders of convergence. (CB) system, nonstandard Galerkin semidiscretization, $\eta_h \in S_h^3$, $u_h \in S_{h,0}^4$, uniform mesh.

problem but now discretized so that $\eta_h \in S_h^3$, i.e. in the space of C^1 quadratic splines, and $u_h \in S_{h,0}^4$, i.e. in

the space of cubic splines that vanish at $x = 0$ and at $x = 1$. We observe e.g. that the associated L^2 -rates of convergence for this higher order pair of subspaces are, as expected, equal to 3 for η and 4 for u . The (SCB) system gave the same results.

The error analysis in paragraphs 5.1 and 5.2 depended strongly on the assumption of uniform mesh. In the case of nonuniform mesh we expect some order reduction. For example, Table 5.3 shows the L^2 errors and orders of convergence that we obtained for (CB) by approximating (η_h, u_h) in $S_h^2 \times S_{h,0}^3$ and using for both subspaces the exact solution and the nonuniform mesh with which Table 2.3(a) was produced. *Both* rates are apparently reduced by one now.

N	η	order	u	order
80	0.1372(-2)		0.3879(-5)	
160	0.6849(-3)	1.002	0.9569(-6)	2.019
240	0.4564(-3)	1.001	0.4241(-6)	2.007
320	0.3422(-3)	1.000	0.2383(-6)	2.004
400	0.2738(-3)	1.000	0.1524(-6)	2.002
480	0.2281(-3)	1.000	0.1058(-6)	2.002
560	0.1955(-3)	1.000	0.7775(-7)	2.001
640	0.1711(-3)	1.000	0.5952(-7)	2.001
720	0.1509(-3)	1.000	0.4702(-7)	2.001

TABLE 5.3. L^2 -errors and orders of convergence. (CB) system, nonstandard Galerkin semidiscretization, $\eta_h \in S_h^2$, $u_h \in S_{h,0}^3$, quasiuniform mesh with $\frac{\max h_i}{\min h_i} = 1.5$

6. REMARKS ON STANDARD GALERKIN METHODS FOR RELATED HYPERBOLIC PROBLEMS

We conclude the paper with a section of remarks on the application of standard Galerkin methods on some simple first-order hyperbolic problems. Our aim is to show examples of such problems in which the techniques previously developed in this paper may be applied to some advantage.

6.1. Initial-boundary-value problem for a single hyperbolic equation. For $T > 0$ consider the initial-boundary-value problem

$$\begin{aligned} \eta_t + \eta_x &= 0, & 0 \leq x \leq 1, \quad 0 \leq t \leq T, \\ \eta(x, 0) &= \eta_0(x), & 0 \leq x \leq 1, \\ \eta(0, t) &= 0, & 0 \leq t \leq T. \end{aligned} \tag{6.1}$$

It is clear that if $\eta_0 \in C^k$, for integer $k \geq 0$, and $\eta_0^{(j)}(0) = 0$, $j = 0, \dots, k$, then (6.1) possesses a unique solution η given for $(x, t) \in [0, 1] \times [0, T]$ by

$$\eta(x, t) = \begin{cases} \eta_0(x - t) & , \quad x > t, \\ 0 & , \quad x < t. \end{cases}$$

The solution is classical if $k \geq 1$ and, for example, has the property that $\eta(\cdot, t) \in C^k$ for all $t \in [0, T]$. For integer $N \geq 2$ and $h = 1/N$ we consider the uniform mesh given by $x_i = ih$, $i = 0, 1, \dots, N$, and put $I_j = [x_{j-1}, x_j]$. We let

$$\mathring{S}_h^2 = \{\phi \in C[0, 1] : \phi|_{I_j} \in \mathbb{P}_1, \quad 1 \leq j \leq N, \quad \phi(0) = 0\},$$

and denote by $\{\phi_j\}_{j=1}^N$ the basis of \mathring{S}_h^2 with the property that $\phi_i(x_j) = \delta_{ij}$. The standard Galerkin method for (6.1) may be then defined as follows: We seek $\eta_h : [0, T] \rightarrow \mathring{S}_h^2$, such that

$$\begin{aligned} (\eta_{ht}, \phi) + (\eta_{hx}, \phi) &= 0, \quad \forall \phi \in \mathring{S}_h^2, \quad 0 \leq t \leq T, \\ \eta_h(0) &= P\eta_0, \end{aligned} \tag{6.2}$$

where P is the L^2 -projection operator onto \mathring{S}_h^2 . Obviously, (6.2) has a unique solution that satisfies e.g. $\|\eta_h(t)\| \leq \|\eta_h(0)\|$, $t \in [0, T]$. Recall that we have $\|v - Pv\| \leq Ch^2\|v\|_2$ for $v \in H^2$ with $v(0) = 0$, and $\|v - Pv\|_\infty \leq Ch^2\|v\|_{2,\infty}$ for $v \in C^2$ with $v(0) = 0$. Our aim is to prove an optimal-order L^2 error estimate for the solution of (6.2), thus extending the result of Dupont, [20], for the periodic initial-value problem for

$\eta_t + \eta_x = 0$ to the case of the initial-boundary-value problem at hand. To do this, we first prove a lemma along the lines of the proof of Lemma 5.6.

Lemma 6.1. *Suppose that v is C^2 and piecewise C^3 on $[0, 1]$ and satisfies $v(0) = v'(0) = v''(0) = 0$. Then, if $\rho = v - Pv$ and $\zeta \in \mathring{S}_h^2$ is such that*

$$(\zeta, \phi) = (\rho, \phi') \quad \forall \phi \in \mathring{S}_h^2,$$

we have $\|\zeta\| \leq Ch^2 \|v^{(3)}\|_\infty$.

Proof. We will first show, as in Lemma 5.6, that if $\varepsilon_i = \int_{I_i} \rho dx$, $1 \leq i \leq N$, then $\varepsilon_i = O(h^4)$. Let $Pv = \sum_{j=1}^N c_j \phi_j$. Then, since $(Pv, \phi_i) = (v, \phi_i)$, $1 \leq i \leq N$, we have $Gc = b$, where $G_{ij} = (\phi_j, \phi_i)$, $1 \leq i, j \leq N$, is the $N \times N$ tridiagonal matrix with $G_{ii} = 2h/3$, $1 \leq i \leq N-1$, $G_{NN} = h/3$, and $G_{ij} = h/6$ if $|i-j| = 1$, and $b_i = (v, \phi_i)$, $1 \leq i \leq N$. Hence,

$$\varepsilon_i = \int_{I_i} \rho dx = \int_{I_i} v dx - \gamma_i, \quad 1 \leq i \leq N, \quad (6.3)$$

where

$$\gamma_i = \int_{I_i} Pvd x = \begin{cases} \frac{hc_1}{2} & \text{if } i = 1, \\ \frac{h(c_{i-1} + c_i)}{2} & \text{if } 2 \leq i \leq N. \end{cases} \quad (6.4)$$

The equations of the system $Gc = b$ are

$$\begin{aligned} \frac{h}{6}(4c_1 + c_2) &= b_1, \\ \frac{h}{6}(c_{i-1} + 4c_i + c_{i+1}) &= b_i, \quad 2 \leq i \leq N-1, \\ \frac{h}{6}(c_{N-1} + 2c_N) &= b_N, \end{aligned}$$

and may be rewritten as

$$\begin{aligned} 3 \cdot \frac{hc_1}{2} + \frac{h(c_1 + c_2)}{2} &= 3b_1, \\ \frac{hc_1}{2} + 4 \cdot \frac{h(c_1 + c_2)}{2} + \frac{h(c_2 + c_3)}{2} &= 3(b_1 + b_2), \\ \frac{h(c_{i-2} + c_{i-1})}{2} + 4 \cdot \frac{h(c_{i-1} + c_i)}{2} + \frac{h(c_i + c_{i+1})}{2} &= 3(b_{i-1} + b_i), \quad 3 \leq i \leq N-1, \\ \frac{h(c_{N-2} + c_{N-1})}{2} + 5 \cdot \frac{h(c_{N-1} + c_N)}{2} &= 3(b_{N-1} + 2b_N). \end{aligned}$$

Therefore, from (6.4) we conclude that

$$\Gamma\gamma = \beta, \quad (6.5)$$

where $\gamma = (\gamma_1, \dots, \gamma_N)^T$, Γ is the $N \times N$ tridiagonal matrix with $\Gamma_{11} = 3$, $\Gamma_{ii} = 4$, $2 \leq i \leq N-1$, $\Gamma_{NN} = 5$ and $\Gamma_{ij} = 1$ if $|i-j| = 1$, and $\beta = (\beta_1, \dots, \beta_N)^T$ with $\beta_1 = 3b_1$, $\beta_i = 3(b_{i-1} + b_i)$, $2 \leq i \leq N-1$, and $\beta_N = 3(b_{N-1} + 2b_N)$. Hence, from (6.3) and (6.5) we have that

$$\Gamma\varepsilon = r, \quad (6.6)$$

where $\varepsilon = (\varepsilon_1, \dots, \varepsilon_N)^T$, and $r = (r_1, \dots, r_N)^T$ is given by

$$\begin{aligned} r_1 &= 3 \int_0^h v dx + \int_h^{2h} v dx - 3b_1, \\ r_i &= \int_{x_{i-2}}^{x_{i-1}} v dx + 4 \int_{x_{i-1}}^{x_i} v dx + \int_{x_i}^{x_{i+1}} v dx - 3(b_{i-1} + b_i), \quad 2 \leq i \leq N-1, \\ r_N &= \int_{x_{N-2}}^{x_{N-1}} v dx + 5 \int_{x_{N-1}}^{x_N} v dx - 3(b_{N-1} + 2b_N). \end{aligned}$$

For r_1 we have

$$r_1 = 3 \int_0^h v(1 - \frac{x}{h}) dx + \int_h^{2h} v(\frac{3x}{h} - 5) dx,$$

and by our hypothesis on v , using Taylor's theorem with remainder in integral form, we have

$$|r_1| \leq Ch^4 \|v^{(3)}\|_{L^\infty[0, 2h]}. \quad (6.7)$$

As in the proof of Lemma 5.6 we may show, using Taylor's theorem with remainder in integral form, that

$$|r_i| \leq Ch^4 \|v^{(3)}\|_\infty, \quad 2 \leq i \leq N. \quad (6.8)$$

Writing now the matrix Γ in the form $\Gamma = 4(I - E)$, we see that E is the a $N \times N$ tridiagonal matrix with $\|E\|_\infty = 1/2$. Hence Γ is invertible, $\|\Gamma^{-1}\|_\infty \leq 1/2$, and (6.6)-(6.8) give

$$\max_{1 \leq i \leq N} |\varepsilon_i| \leq Ch^4 \|v^{(3)}\|_\infty. \quad (6.9)$$

Since $(\zeta, \phi_i) = (\rho, \phi'_i)$, $1 \leq i \leq N$, we conclude from (6.9) that

$$\max_{1 \leq i \leq N} |(\zeta, \phi_i)| \leq Ch^3 \|v^{(3)}\|_\infty. \quad (6.10)$$

It is straightforward to check now that Lemma 2.1(i) and (ii) hold for \hat{S}_h^2 as well, *mutatis mutandis*. The conclusion of the Lemma follows. \square

Proposition 6.1. *Suppose that $\eta_0 \in C^3$ with $\eta_0(0) = \eta'_0(0) = \eta''_0(0) = 0$. Then, if η_h is the solution of the semidiscrete problem (6.2) we have that*

$$\max_{0 \leq t \leq T} \|\eta(t) - \eta_h(t)\| \leq C(T)h^2 \max_{0 \leq t \leq T} \|\eta\|_{3,\infty}.$$

Proof. Let $\rho = \eta - P\eta$, $\theta = P\eta - \eta_h$. Then, for $0 \leq t \leq T$

$$(\theta_t, \phi) + (\theta_x, \phi) + (\rho_x, \phi) = 0 \quad \forall \phi \in \hat{S}_h^2.$$

Putting in this $\phi = \theta$ we obtain, using integration by parts, that

$$\frac{1}{2} \frac{d}{dt} \|\theta(t)\|^2 + \frac{1}{2} \theta^2(1, t) = -\rho(1, t) \theta(1, t) + (\rho, \theta_x). \quad (6.11)$$

Now, $|\rho(1, t)| \leq \|\rho\|_\infty \leq Ch^2 \|\eta\|_{2,\infty}$ and our hypotheses on η_0 imply that $\eta(\cdot, t) \in C^2$ and piecewise C^3 for $0 \leq t \leq T$ and satisfies $\partial_x^j \eta(0, t) = 0$ for $j = 0, 1, 2$, $0 \leq t \leq T$. Therefore, by Lemma 6.1 we have, if $(\zeta, \phi_i) = (\rho, \phi'_i)$, $1 \leq i \leq N$, that $(\rho, \theta_x) = (\zeta, \theta) \leq Ch^2 \|\eta\|_{3,\infty} \|\theta\|$. Hence, by the arithmetic-geometric mean inequality we obtain from (6.11) that

$$\frac{1}{2} \frac{d}{dt} \|\theta(t)\|^2 \leq Ch^4 \|\eta(t)\|_{3,\infty}^2 + \|\theta(t)\|^2, \quad 0 \leq t \leq T.$$

Therefore, Gronwall's Lemma and (6.2) give the conclusion of the Proposition. \square

In Table 6.1 we show the results of a numerical experiment that we performed to investigate the influence of the degree of compatibility of the initial data η_0 at $x = 0$ on the order of convergence of the L^2 - error of the standard Galerkin semidiscretization of (6.1) using the space \hat{S}_h^2 for the space discretization and $P\eta_0$

	$x \exp(x)$		$x^2 \exp(x)$		$x^3 \exp(x)$		$x^4 \exp(x)$	
N	L^2 -error	order	L^2 -error	order	L^2 -error	order	L^2 -error	order
50	0.9811(-3)		0.3014(-3)		0.4786(-3)		0.6765(-3)	
100	0.4436(-3)	1.049	0.7583(-4)	1.991	0.1204(-3)	1.991	0.1705(-3)	1.989
150	0.2891(-3)	1.056	0.3379(-4)	1.993	0.5360(-4)	1.995	0.7597(-4)	1.994
200	0.2145(-3)	1.039	0.1904(-4)	1.993	0.3018(-4)	1.997	0.4279(-4)	1.996
250	0.1699(-3)	1.043	0.1221(-4)	1.993	0.1933(-4)	1.997	0.2740(-4)	1.997
300	0.1407(-3)	1.034	0.8489(-5)	1.993	0.1343(-4)	1.998	0.1904(-4)	1.997
350	0.1199(-3)	1.037	0.6244(-5)	1.992	0.9868(-5)	1.998	0.1399(-4)	1.998
400	0.1045(-3)	1.030	0.4786(-5)	1.992	0.7557(-5)	1.998	0.1072(-4)	1.998
450	0.9255(-4)	1.033	0.3785(-5)	1.992	0.5972(-5)	1.999	0.8469(-5)	1.998
500	0.8305(-4)	1.028	0.3069(-5)	1.991	0.4838(-5)	1.999	0.6861(-5)	1.998
550	0.7528(-4)	1.030	0.2538(-5)	1.991	0.3999(-5)	1.999	0.5671(-5)	1.999
600	0.6885(-4)	1.026	0.2135(-5)	1.990	0.3360(-5)	1.999	0.4766(-5)	1.999

TABLE 6.1. L^2 errors and convergence rates. Standard Galerkin semidiscretization for problem (6.1) with piecewise linear, continuous functions and four choices of $\eta_0(x)$.

as initial data. We considered (6.1) with $T = 0.5$ taking successively $\eta_0(x)$ equal to $x \exp(x)$, $x^2 \exp(x)$, $x^3 \exp(x)$ and $x^4 \exp(x)$. (The temporal discretization was effected with the Crank-Nicolson implicit scheme with time step $k = h/3$ in all cases; this time step was sufficiently small to ensure that the temporal error was about two orders of magnitude less than the spatial error in all cases.) The L^2 errors and associated rates of convergence of the numerical scheme at $t = T = 0.5$ for diminishing $h = 1/N$ are shown in Table 6.1.

Recall that if $\eta_0(x) = x^k \exp(x)$, $k = 1, 2, 3, \dots$, the solution $\eta(x, t)$ of (6.1) is in $C^{k-1}(0, 1)$ (and piecewise C^∞) for all $t \in [0, T]$. The L^2 orders of convergence turn out to be practically equal to 2 for $k = 4$ and $k = 3$ in agreement with the error estimate that was proved. The rate is apparently converging to one in the case $k = 1$ in which (6.1) possesses a generalized, piecewise smooth solution, while in the borderline case $k = 2$ the convergence rate has not stabilized. (When we increased the number of spatial intervals N , we observed that this rate fell steadily and was e.g. equal to 1.9826 at $N = 1600$.)

6.2. A hyperbolic equation with a variable coefficient that vanishes at the boundary. Consider the problem

$$\begin{aligned} \eta_t + (u\eta)_x &= 0, \quad 0 \leq x \leq 1, \quad 0 \leq t \leq T, \\ \eta(x, 0) &= \eta_0(x), \quad 0 \leq x \leq 1, \end{aligned} \tag{6.12}$$

where u is a given smooth function that vanishes at $x = 0$ and at $x = 1$. No boundary conditions are imposed on η . The problem is well posed and may be solved by the method of characteristics. Uniqueness of solutions follows easily: If we multiply the p.d.e. in (6.12) by η and integrate over $[0, 1]$ by parts we see that for $0 \leq t \leq T$

$$\frac{1}{2} \frac{d}{dt} \|\eta\|^2 + \frac{1}{2} \int_0^1 \eta^2 u_x dx = 0,$$

from which, by Gronwall's Lemma, there follows that

$$\|\eta(t)\| \leq C \|\eta_0\|, \quad 0 \leq t \leq T,$$

where $C = C(T, \max_{0 \leq t \leq T} \|u_x\|_\infty)$.

An optimal-order L^2 error estimate for the standard Galerkin semidiscretization in S_h^2 on a uniform mesh may be proved for this problem using the methods previously developed in this paper. (This was first pointed out in [3].) The semidiscrete problem consists in finding $\eta_h : [0, T] \rightarrow S_h^2$ so that

$$\begin{aligned} (\eta_{ht}, \phi) + ((u\eta_h)_x, \phi) &= 0, \quad \forall \phi \in S_h^2, \quad 0 \leq t \leq T, \\ \eta_h(0) &= P\eta_0, \end{aligned} \tag{6.13}$$

where P is the L^2 projection onto S_h^2 .

Proposition 6.2. *Suppose that $u \in C_0^2$ and that the solution of (6.12) satisfies $\eta \in C(0, T; C^4)$. Then*

$$\max_{0 \leq t \leq T} \|\eta - \eta_h\|_\infty \leq Ch^2.$$

Proof. Putting $\theta = P\eta - \eta_h$, $\rho = \eta - P\eta$ we have for $0 \leq t \leq T$ that

$$(\theta_t, \phi) + ((u\theta)_x, \phi) + ((u\rho)_x, \phi) = 0 \quad \forall \phi \in S_h^2. \tag{6.14}$$

Now, by Lemma 5.1, if for each $t \in [0, T]$, $\zeta \in S_h^2$ is defined by

$$(\zeta, \phi) = ((u\rho)_x, \phi) \quad \forall \phi \in S_h^2,$$

we have that $\|\zeta\| \leq Ch^3$. Therefore, putting $\phi = \theta$ in (6.13) and using integration by parts we see that

$$\|\theta\| \leq Ch^3.$$

We conclude that $\|\eta - \eta_h\|_\infty \leq \|\theta\|_\infty + \|\rho\|_\infty \leq Ch^{-1/2} \|\theta\| + \|\rho\|_\infty \leq Ch^2$. \square

This observation may prove useful in the error analysis of first-order problems with advection terms of the form $(u\eta)_x$, where u satisfies Dirichlet boundary conditions. However, as we saw in Section 2 in the case of the Boussinesq systems under study the presence of the linear u_x term in the first p.d.e. of (CB) or (SCB) is the obstacle that prevents achieving optimal-order estimates. For example, examining the proof of Theorem 2.1 we note that if the u_x term was missing, then the terms (ξ_x, ϕ) and (σ_x, ϕ) would not be present in (2.21) and, consequently, by appeal to Lemma 2.2 with $w(0) = w(1) = 0$, the estimates of the terms in (2.24) would lead to an $O(h^2)$ bound in the right-hand side of (2.25).

6.3. Numerical experiments with nonuniform meshes. When we used a *quasiuniform mesh* in the standard Galerkin semidiscretization (6.2) of the initial-boundary-value problem (6.1), numerical results (not shown here) indicating that the L^2 error $\|\eta - \eta_h\|$ is of $O(h)$; of the same order of accuracy is of course the upper bound of the L^2 error predicted in a straightforward manner by the theory.

The same differences between the cases of uniform and nonuniform meshes apparently also persist in the case of first-order hyperbolic *systems*. Consider, for example, the following initial-boundary-value problem for the nonhomogeneous wave equation written in first-order system form:

$$\begin{aligned} \eta_t + u_x &= f, & 0 \leq x \leq 1, \quad 0 \leq t \leq T, \\ u_t + \eta_x &= g, \\ \eta(x, 0) &= \eta_0(x), \quad u(x, 0) = u_0(x), \quad 0 \leq x \leq 1, \\ \eta(0, t) &= 0, \quad u(1, t) = 0, \quad 0 \leq t \leq T. \end{aligned} \tag{6.15}$$

For the purposes of the numerical experiment we took as exact solution of (6.15) the pair of functions $\eta(x, t) = x \exp(x(t+1))$, $u(x, t) = (x-1) \exp(xt)$ with appropriate initial conditions and right-hand sides. We discretized the problem in space by the standard Galerkin method using piecewise linear continuous elements on a uniform mesh with meshlength $h = 1/N$ and also on a quasiuniform mesh with $\Delta x = 1.6/N$, $h_{2i-1} = 0.75\Delta x$, $h_{2i} = 0.5\Delta x$, $1 \leq i \leq N/2$. For the temporal discretization we used the classical, fourth-order explicit Runge-Kutta scheme with $k = h$ and $k = \Delta x$, respectively. As initial values $\eta_h(0)$, $u_h(0)$ we took the interpolants of η_0 and u_0 . The L^2 errors at $T = 0.4$ and the corresponding orders of convergence are shown, for increasing N , in Table 6.2(a) in the case of the quasiuniform mesh and in Table 6.2(b) for the uniform mesh. The experiment strongly suggests that the L^2 errors are of $O(h^2)$ for a uniform mesh and of $O(\Delta x)$ for quasiuniform.

N	$L^2\text{-errors}(\eta)$	$order$	$L^2\text{-errors}(u)$	$order$
80	0.1577(-2)		0.1607(-2)	
160	0.7813(-3)	1.013	0.8002(-3)	1.006
240	0.5194(-3)	1.007	0.5327(-3)	1.003
320	0.3891(-3)	1.004	0.3992(-3)	1.002
400	0.3110(-3)	1.004	0.3193(-3)	1.002
480	0.2590(-3)	1.003	0.2660(-3)	1.001
560	0.2219(-3)	1.002	0.2279(-3)	1.001
640	0.1941(-3)	1.002	0.1994(-3)	1.001
720	0.1725(-3)	1.002	0.1772(-3)	1.001

(a)

N	$L^2\text{-errors}(\eta)$	$order$	$L^2\text{-errors}(u)$	$order$
80	0.8117(-4)		0.3932(-4)	
160	0.2029(-4)	2.000	0.9825(-5)	2.001
240	0.9021(-5)	2.000	0.4369(-5)	1.999
320	0.5074(-5)	2.000	0.2457(-5)	2.000
400	0.3248(-5)	2.000	0.1573(-5)	1.999
480	0.2255(-5)	2.000	0.1092(-5)	2.000
560	0.1657(-5)	2.000	0.8026(-6)	2.000
640	0.1269(-5)	2.000	0.6145(-6)	2.000
720	0.1002(-5)	2.000	0.4856(-6)	2.000

(b)

TABLE 6.2. L^2 -errors and orders of convergence, problem (6.15). Standard Galerkin discretization with piecewise linear, continuous functions. (a): quasiuniform mesh (b): uniform mesh.

As a final remark, we consider the following initial-boundary value problem of a hyperbolic system with a viscous term

$$\begin{aligned} \eta_t + u_x &= 0, \\ u_t + \eta_x - u_{xx} &= 0, & 0 \leq x \leq 1, \quad 0 \leq t \leq T, \\ \eta(x, 0) &= \eta_0(x), \quad u(x, 0) = u_0(x), \quad 0 \leq x \leq 1, \\ u(0, t) &= 0, \quad u(1, t) = 0, \quad 0 \leq t \leq T, \end{aligned} \tag{6.16}$$

and discretize it by the standard Galerkin method. We obtain similar results to those of Section 2 in which the dispersive term u_{xxt} is present instead of u_{xx} . (Cf. Remark 2.3) In a numerical example we took a nonhomogeneous version of (6.16) with exact solution $\eta(x, t) = \exp(2t)(\cos(\pi x) + x + 2)$, $u(x, t) = \exp(xt)(\sin(\pi x) + x^3 - x^2)$ and discretized the problem in space with the standard Galerkin method using a uniform mesh with $h = 1/N$ and the quasiuniform mesh of the previous example. We integrated up to $T = 0.5$ with the classical, fourth-order, explicit Runge-Kutta scheme with $k = h^2/25$ in the uniform mesh and $k = (\Delta x)^2/25.6$ in the nonuniform mesh case. The errors and L^2 convergence rates at $T = 0.5$ shown in Table 6.3 suggest that the L^2 errors for η and u are of $O(\Delta x)$ and $O(\Delta x)^2$, respectively, in the quasiuniform mesh case, and of $O(h^{3/2})$ and $O(h^2)$ in the uniform case, exactly as in the linear dispersive case.

N	$L^2\text{-errors}(\eta)$	$order$	$L^2\text{-errors}(u)$	$order$
20	0.2013(-1)		0.1771(-2)	
40	0.1004(-1)	1.003	0.4434(-3)	1.998
60	0.6696(-2)	0.999	0.1971(-3)	1.999
80	0.5023(-2)	0.999	0.1109(-3)	1.999
100	0.4019(-2)	0.999	0.7099(-4)	2.000
120	0.3350(-2)	0.999	0.4930(-4)	2.000
140	0.2872(-2)	0.999	0.3622(-4)	2.000
160	0.2513(-2)	0.999	0.2774(-4)	2.000

(a)

N	$L^2\text{-errors}(\eta)$	$order$	$L^2\text{-errors}(u)$	$order$
20	0.3605(-2)		0.1396(-2)	
40	0.1149(-2)	1.649	0.3487(-3)	2.001
60	0.6011(-3)	1.598	0.1550(-3)	2.000
80	0.3823(-3)	1.574	0.8716(-4)	2.000
100	0.2700(-3)	1.559	0.5578(-4)	2.000
120	0.2035(-3)	1.549	0.3873(-4)	2.000
140	0.1605(-3)	1.542	0.2846(-4)	2.000
160	0.1307(-3)	1.537	0.2179(-4)	2.000

(b)

TABLE 6.3. L^2 -errors and orders of convergence, problem (6.16). Standard Galerkin discretization with piecewise linear, continuous functions. (a): quasiuniform mesh (b): uniform mesh.

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